92

A
COMPLEAT SYSTEM
OF
OPTICKS

In Four BOOKS, viz.

A Popular, a Mathematical, a Mechanical, and a Philosophical Treatise.

To which are added

REMARKS upon the Whole.

BY

ROBERT SMITH LL.D.

Professor of Astronomy and Experimental Philosophy at CAMBRIDGE, and Master of Mechanicks to his MAJESTY.

Quid tam mirabile, quam particulam corporis quandam ita fabricatam esse, ut ejus opera animal sentiat procul positorum corporum figuram, positum, motum quemlibet, distantiam; idque etiam cum colorum varietate, quo distinctius ea dignosceret? Nihil est, in quo manifestius Geometriæ artem Deus exercuerit. Hugenii Cosmotheoros. p. 40.

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Printed for the AUTHOR, and sold there by *Cornelius Crownfield*, and at *London* by *Stephen Austen* at the *Angel* and *Bible* in *St. Paul's Church-yard*, and *Robert Dodsley* at *Tully's Head* in *Pall-Mall*. MDCCXXXVIII.

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SIR,

OF all the incidents in my life there is none I reflect upon with greater pleaſure, than the happineſs of falling into your acquaintance at CAMBRIDGE. There, among the many agreeable hours I had the honour of paſſing in your company, you may remember the converſation ſometimes turned upon Opticks, a Science you knew I had taken ſome pains in: And the curioſity you then diſcovered, in relation to ſome

points

points that had not been ſufficiently explained by former writers, was the principal motive with me to take them more particularly into conſideration.

BUT as our acquaintance grew by degrees into a ſettled friendſhip, I became deſirous of giving you ſome publick teſtimony of the reſpect and affection I had for you; to which end I began to collect together and reviſe my ſcattered papers, and to think of preparing them for the Preſs. And ſo ſollicitous was I to produce ſomething not unworthy of the Patronage you were then pleaſed to grant me, that I attended too little to the good old rule, *Quid valeant humeri*, and projected a deſign much too large for the health and leiſure I had to proſecute it.

BY the want of theſe, the work has been retarded for ſo many years, that in the more active and conſpicuous ſcene of life you have ſince entered upon, wherein you appear with ſo much honour to yourſelf and ſo much pleaſure to your friends, when your former thoughts upon this ſubject muſt neceſſarily have been long ſince diſcontinued and laid aſide, I am ſenſible you can hardly have time to look into, or think of

what

what in the days of your Academical leiſure would have been no diſagreeable entertainment.

To you however I muſt beg leave to addreſs it, becauſe, of how little uſe or pleaſure ſoever it may now be to you, to you the Preſent of right belongs. 'Twas you that ſet me firſt to work; 'twas you that procured me leiſure in time to finiſh it; 'twas my affection for you and that alone, that gave me ſtrength and ſpirit to go through with it.

For in the weak and unſettled ſtate of health I have been in for many years, the common motives to ſuch undertakings, as the importunity of acquaintance, the hopes of fame, or even the deſire of promoting uſeful knowledge, could never have been ſtrong enough to carry me through ſo large and difficult a Work, had I not been animated by a warmer affection ariſing from the amiable qualities of the beſt of Friends, improved by a long experience and obſervation of them, and ſtill farther heightened by a train of favours and honours received from his hand and heart.

THESE effects of your Friendſhip for me are ſo much to my reputation, that you muſt pardon my vanity in taking this publick notice of them. Were the Treatiſe I preſent you with at all anſwerable to my ambition, it ſhould be a laſting Monument of the Gratitude of,

SIR,

Your moſt affectionate Friend,

and humble Servant,

ROBERT SMITH.

PREFACE.

THE first of the four Books, or the Popular Treatise, though partly intended as an Introduction to the rest, is chiefly designed for the use of those who would know something of Opticks, but want the preparatory learning that is necessary for a thorough acquaintance with that Science. With this view I have here avoided all Geometrical Demonstrations, and instead thereof have substituted that more entertaining and looser sort of proof, that may be drawn from experiment only; and the experiments I have contrived for that end, are not only easy to be understood, but may, if the Reader pleases, be tryed with very little trouble or apparatus

By this means one who uses a little application, may find himself master of no inconsiderable part of the doctrine of Opticks, which is here explained in such a manner, as I hope may be easy to all, and yet not tedious to more skilful Readers, who perhaps may find here and there something not unworthy of their notice.

Another advantage designed for the Readers above-mentioned is, that this Popular Treatise well understood, will be abundantly sufficient to conduct them through many curious pieces contained in the Remarks, and even through the whole third and fourth Books; especially if their heads be a little turned towards mechanical matters, and be furnished with some of the first and easiest principles of Astronomy.

In the third Book, besides some curious collections belonging to the art of grinding Glasses, drawn up and communicated by the late Hon. Samuel Molyneux *Esq; I have given a full description of a compleat Set of Optical and Astronomical Instruments, according to the latest and best improvements; together with particular explanations of their several uses, when applyed to the purposes of Astronomy, Geography, Navigation, Levelling, and other useful Arts.*

And in the fourth Book I have given a compleat history and collection of Telescopical Discoveries in the heavens, selected from a great variety of books, memoirs, and observations of the best Astronomers. To which I intended to have added a like collection of Microscopical Discoveries, but found my Work was grown too large without it.

Thus

Thus I have endeavoured to lay open a large field of Knowledge, even to persons very moderately prepared for this sort of Learning.

Those that are better versed in Euclid's *Elements, after reading the four first chapters of my second Book, which contain the Geometrical Elements of the whole Work, demonstrated in a short and easy manner, may then proceed to such parts of it as they are most inclined to. For I have endeavoured to make all the Chapters in the four Books, and the pieces in the Remarks, as much independent on one another as I could; chusing rather to repeat a few matters in the order and form in which they occasionally presented themselves, than to trouble my readers to recur to them elsewhere in a different dress: not to mention that different views of the same truths are seldom disagreeable to men of taste, and are equally useful to beginners with the writings of different authors upon the same subject.*

As I generally demonstrate every thing from the first principles, the Remarks are seldom explanatory of any thing in the four Books, but are chiefly additions to them; as containing the History of Inventions and their improvements, dissertations on different opinions in difficult points, confutations of errors, collections of hints, observations and queries, subjects enlarged upon or reconsidered in a different manner, both physical and mathematical, as curious and valuable as any in the four Books: and as some of these pieces are pretty long, I should have printed them with a letter somewhat larger, had the press been furnished with it.

As the general contents of the whole Work will sufficiently appear by the Table of the Chapters, and the particulars by the Index, I might here conclude; but as the Reader may expect some account of what is to be found in this Work, either of new discovery, or of improvement upon what has been delivered by other writers, I shall here mention some of the principal matters.

1. *The determination of the Focus of a pencil of reflected rays, after falling directly or obliquely on any number of plane or spherical surfaces, or of refracted rays, after passing directly or obliquely through any number of lens's of any thickness, or through different mediums having plane or spherical surfaces, is made more general and easy; by reducing it in all these cases to the like simple proportion, that determines the focus of a pencil after falling directly upon a single surface: and even in the Popular Treatise, I have given a plainer and fuller idea of the positions and relative motions of conjugate focus's along the axis of the glasses, than what I could meet with in Books of Opticks.*

2. *The*

2. *The determination of the Aberrations of reflected and refracted rays from the geometrical focus, caused by their different refrangibility and by the sphericalness of the figure of the surfaces, is made easier in simple cases, is treated more copiously in different ways, is made more general, and applied to more complex constructions of* Optical Instruments *than heretofore.*

3. *Hence after demonstrating the known rules for proportioning the lengths, apertures and eye-glasses of reflecting and refracting Telescopes, the theory of a reflecting Microscope, having a concave spherical speculum and a convex eye-glass, as proposed by Sir* Isaac Newton, *is fully considered; and rules are given for the improvement of this microscope, as far as its construction will admit.*

4. *The rules delivered by Mr.* Huygens *for the improvement of refracting Microscopes, both single and double, are also considered and demonstrated But since the magnifying powers of all sorts of microscopes yet extant, are limited by the insuperable difficulty of truly figuring a lens or speculum of so small a size, as their constructions require for magnifying more than ordinary; in considering how to remove that difficulty, I found out a construction with two spherical speculums and a convex eye-glass, wherein that excessive smallness is not necessary. And after due consideration of the theory here described, and an Essay of it made by Mr.* Short *of* Edinburgh, *I am of opinion, if sufficient care be taken in the practical part, to follow the directions and the Table of Measures here given, that this microscope may far excel all others yet extant: I mean in magnifying transparent objects more than usual; which is generally sufficient, because the particles of all substances will become transparent when rendred sufficiently small, and these are the particles that we want to see magnified more than ordinary.*

5. *Mr.* Gregorie *proposed to construct his Telescope with speculums figured according to the conick sections, which being impracticable, the necessity of using spherical speculums has rendred the theory much more complex, on account of the aberrations of the rays. His telescope and that of Mr.* Cassegrain *I have considered very minutely, and have given a solution of this problem. Having the focal distance of the larger speculum, the angle of vision, the degrees of apparent brightness and distinctness, with which the object shall appear, to construct the telescope. Hence I have calculated a table of the dimensions and magnifying powers of these telescopes.*

6. *To the description of the Binocular Telescope I have added a solution of the most remarkable phænomenon belonging to it, and of two or three more of the same kind relating to double vision.*

7. *To the description and properties of Causticks, I have added in some cases a determination of the density of the rays in their several parts; and have compared the powers of Burning-Glasses of several sorts, one with another; whereby it appears which glasses are the best for casting a strong light upon microscopical objects.*

8. *I have given a more general solution of the known problem for finding the diameters and breadths of Rain-bows; and have composed some propositions to shew the variations of the apparent magnitude, figure and brightness of the Sun, when seen by rays refracted in various angles through spherical bodies; and have considered Sir* Isaac Newton's *thoughts upon Halo's, whose phænomena, he tells us in the preface to his Opticks, he endeavoured to account for, but for want of sufficient observations, left that matter to be farther examined. I have also reduced the mathematical matters belonging to Mr.* Huygens's *theory of Corona's and Parbelia, to a few propositions, and demonstrated the construction of his tables. And have shewn why Corona's or Halo's about the sun and moon, appear not circular but oval.*

9. *This is a natural consequence from our idea of the Sky, whose apparent Figure I have here considered; and taking it for a segment of a spherical surface, as it generally seems to be, have determined the proportion of its altitude to the diameter of its base; and from hence have deduced an adequate solution of the long disputed question, why the Sun, Moon and Constellations appear larger near the horizon, than at higher elevations, and in what proportions; which proportions agree so well to our common conceptions of their different magnitudes at different elevations, as to amount to a physical proof of the truth of this solution; especially as it is applicable to many other phænomena of the same kind, and is confirmed in the Remarks by an experiment made upon a like appearance. I have also offered some reasons, to be farther examined, why the Horizontal Moon appears now and then of a size extraordinary large; and have determined the proportion of moon-light to day-light.*

10. *The causes that suggest our ideas of Distance, and the determination of the Apparent Distance of an object seen in glasses, is another famous inquiry of no small difficulty, upon which much has been written, but with little certainty and satisfaction to the curious. I have therefore considered this point*

in a very particular manner, and have settled it on such a foundation of reason and experience, as, I hope, will admit of no doubt or dispute for the future. And upon the Principle by which I introduce the consideration of apparent distance into geometry, I have not only determined it in vision with any number of glasses, but by the help of Geometrical Places, have shewn its regular variations, while the eye, object, or system of glasses are moving forwards or backwards; and have found the variations so determined to be agreeable to experience.

11. *By the help of the said Principle, and of an admirable Dioptrick Theorem invented by Mr.* Cotes, *I was also enabled to give very general and yet very easy determinations of the Apparent Distance, Magnitude, Situation, Distinctness, Brightness, the greatest Angle of Vision and Visible Area, that is, of all the appearances of an object seen by rays coming from any number of speculums, lens's or mediums having plane or spherical surfaces; and in corollaries from them to deduce the known properties of Telescopes and Microscopes of all sorts; which however are independently demonstrated in other places of the Book and Remarks.*

12. *In farther confirmation of the truth and extent of the Principle abovementioned, I have also applied it to one of the most difficult subjects in Opticks, upon which the best writers have not yet succeeded. It is to determine the Apparent Shape of a large plane object, distorted by too oblique reflections from spherical speculums, or too great refractions through spherical mediums, when viewed either by one eye alone or both; which in some cases alters the appearance very surprizingly, and by the bye accounts for that admirable effect of a large concave speculum in heightening the Relievo of Pictures.*

13. *Lastly, I have drawn up some general Theorems on purpose for computing the diameter of the Image of an object, whether distinctly or indistinctly formed upon the Retina or any surface parallel to it; and for shewing its properties and variations upon varying the distance of the object; also for computing the diameter, and shewing the variations of the section of a single pencil cut by the retina or a surface parallel to it; and for determining the place of one or more refracting surfaces, requisite to transfer the rays from one given focus to another.*

This is a short account of the general matters to be found in this Treatise, besides collections from other authors, whose thoughts and words too I have not scrupled to copy for many pages together, as often as they suited my purpose; and when they did not, those that are conversant in Books of Opticks will find,

that

that what I have borrowed from others, is not the worse for having passed through my hands.

I must now make my acknowledgements to my Friends and Acquaintance for the assistance they have been pleased to give me; viz. *to* Martin Folkes *Esq*; *for his curious remarks on fallacies in vision, on the sun's apparent distance, on the apparent figure of the sky, on the apparent curvity of the sides of long walks and ploughed lands, and the changes of curvity by the observer's motion*; *to* John Hadley *Esq*; *to whom we owe the present perfection of reflecting telescopes, for a full and accurate description of his manner of making the speculums*; *to Mr.* Mac Laurin *Professor of Mathematicks at* Edinburgh, *for his account of some reflecting telescopes made with glass speculums by Mr.* Short *of* Edinburgh; *to Mr.* George Graham *F.R.S. for assisting me in the description of his Astronomical Sector, and of the Mural Arch in the Royal Observatory at* Greenwich; *and to Dr.* Jurin, *for his solution of Mr.* Molyneux's *problem mentioned by Mr.* Locke, *whether a person born blind and made to see when adult, could distinguish a globe from a cube at first sight*; *for his remarks on fallacies of sight, and on the association of ideas*; *for his dissertation on squinting*; *for his experiments to shew how much brighter an object appears to both eyes than to one alone*; *for his solution of some surprizing phænomena in double vision*; *and lastly for his* Essay on distinct and indistinct vision; *which contains so great a variety of new observations, curious discoveries, and difficult points, discussed and determined with so much perspicuity, penetration and judgment, that for a just idea of them the reader must have recourse to the Essay it self. When he has read it and the curious pieces communicated by those other gentlemen, I am sure he will join with me in thanks to their several Authors.*

ERRATUM.

Essay pag. 149. Col. 1. lin. 10. *from the bottom, read*, as the image of the diameter of the silk.

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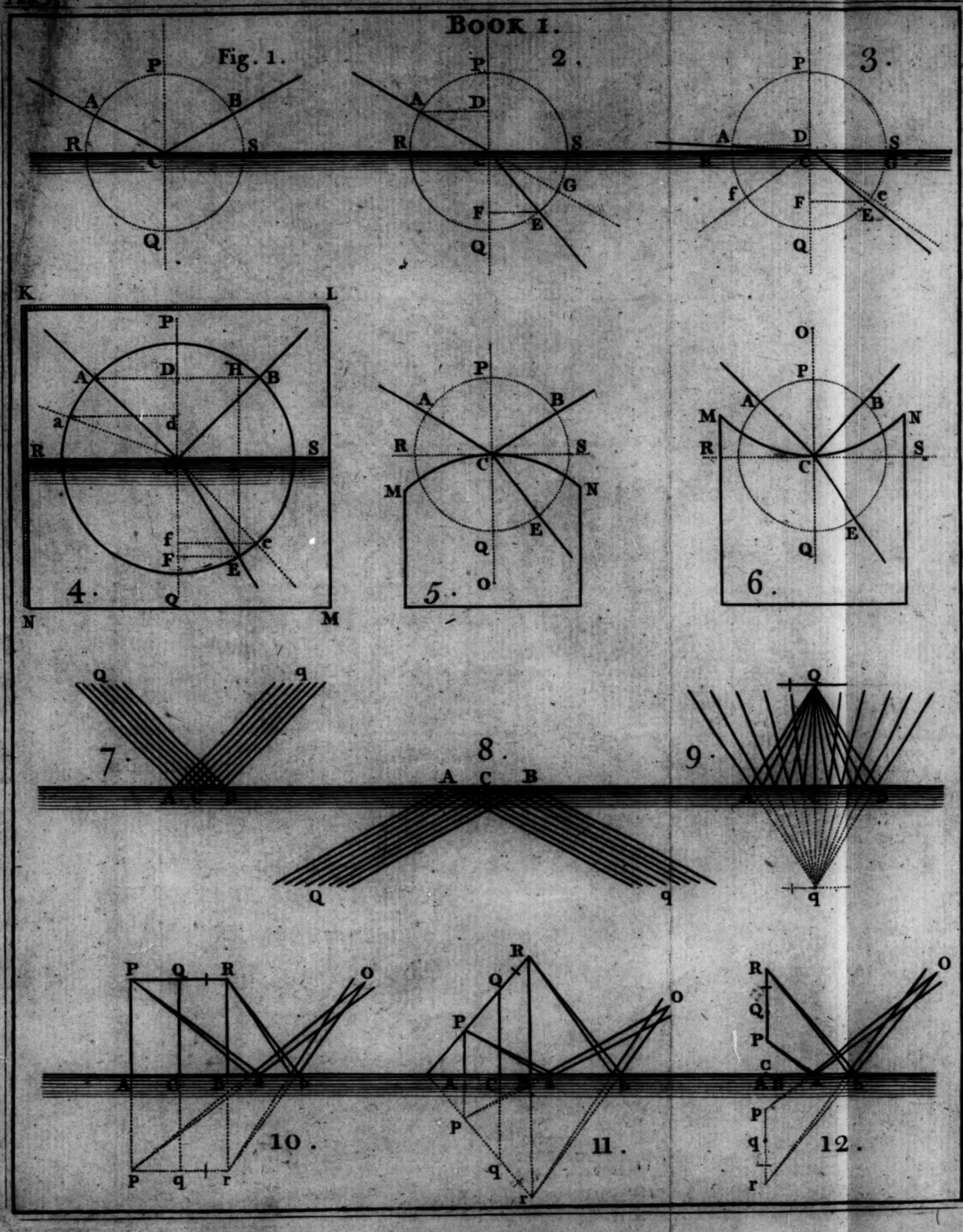
GEORGE R.

GEORGE the Second, by the Grace of God, King of *Great Britain*, *France* and *Ireland*, Defender of the Faith *&c.* To All, to whom these Presents shall come, Greeting. Whereas our Trusty and Well-beloved ROBERT SMITH, Doctor of Laws, Professor of Astronomy and Experimental Philosophy in Our University of *Cambridge*, has humbly represented unto Us, that he has with great Labour and Expence composed, and prepared for Publication, several Philosophical, Optical, Mathematical, and Astronomical Works, to the Copies of which he hath the sole Right and Title; and hath humbly besought Us, to grant him Our Royal Privilege and Licence for the sole Printing and Publishing them, for the Term of Fourteen Years; We, being willing to give due Encouragement to Works of this Nature, that tend to the Advancement and Propagation of Learning, are graciously pleased to condescend to his Request, and do, therefore by these Presents, so far as may be agreeable to the Statute in that Behalf made and provided, grant unto him the said ROBERT SMITH, his Heirs, Executors and Assigns, Our Royal Privilege and Licence, for the sole Printing and Publishing the said Works, for the Term of Fourteen Years, to be computed from the Date hereof; Strictly forbidding and prohibiting all Our Subjects within our Kingdoms and Dominions to Reprint, Abridge or Translate the same, either in the like or any other Volume or Volumes whatsoever; or to import, buy, vend, utter or distribute any Copies of them, reprinted beyond the Seas, during the said Term of Fourteen Years, without the Consent and Approbation of the said ROBERT SMITH, his Heirs, Executors, and Assigns, by Writing under his, or their Hands and Seals first had and obtained, as they, and every of them offending herein, will answer the contrary at their Perils; Whereof the Commissioners and other Officers of Our Customs, the Master, Wardens and Company of Stationers of Our City of *London*, and all Others Our Officers and Ministers, whom it may concern, are to take Notice, that due Obedience be given to Our Pleasure herein signified. Given at Our Court at St. *James*'s the Seventh Day of November One Thousand Seven Hundred and Thirty Seven. In the Eleventh Year of Our Reign.

By His MAJESTY's Command

HOLLES NEWCASTLE.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A COMPLEAT SYSTEM OF OPTICKS.

BOOK I.

A Popular Treatise.

Chap. I.

Concerning Light.

Whoever has considered what a number of properties and effects of light are exactly similar to the properties and effects of bodies of a sensible bulk, will find it difficult to conceive that light is any thing else but very small and distinct particles of matter *: which being incessantly thrown out from shining substances, and every way dispersed by reflection from all others, do impress upon our organs of seeing that peculiar motion, which is requisite to excite in our minds the sensation of light. But for the present purpose it is sufficient to observe that light consists of parts, both successive in the same lines and contemporary in several lines: because in the same place, you may stop that which comes one moment, and let pass that which comes presently after; and at the same time, you may stop it in one place, and let it pass in another. For that part of the light which is stopt cannot be the same with that which is let pass.

Light consists of parts.

2. The least light or part of light, which may be stopt alone without the rest of the light, or propagated alone, or do or suffer any thing alone, which the rest of the light doth not or suffereth not, is called a Ray of light †. That rays of light are straight, is evident enough from the shadows of bodies; or from the appearance

A ray of light what and how considered.

* Newt. Opt. Qu. 29. p. 345. 8°. Edit. † Newton's definition. Opt. p. 1.

of light passing through little holes into a dark room full of dust or smoke; or because bodies cannot be seen through the bore of a bended pipe; or because they cease to be seen by the interposition of other bodies, as the fixt stars by the interposition of the moon and planets; and the parts of the sun by the interposition of the Moon, Mercury or Venus. Rays of light may therefore be represented by straight lines, not Mathematical but Physical, which are described by the motion of the parts or particles of light: and the point which a ray possesses in falling upon any surface may be considered as a Physical Point.

The manner of reflection and refraction of a ray described. Fig. 1.

3. When a ray of light falls obliquely upon a smooth polished surface, it is turned out of its way either by reflection or refraction in the following manner. Imagine the paper upon which this figure is drawn to be perpendicular to the surface of stagnating water, and to cut it in the line *RS*, and that a ray of light, coming in the air along the line *AC*, falls upon *RS* at the point *C*. Then supposing the line *PCQ* to be perpendicular to the surface of the water, if the ray be reflected, or turned back at *C* into the air again, it will describe a straight line *CB*, inclined to the perpendicular *CP* at an angle *PCB* exactly equal to the angle *PCA*.

Fig. 2.

But if the ray that came along *AC* goes into the water at *C*, it will not proceed straight forward, but being refracted or bent at *C*, it will describe another straight line *CE* inclined to the perpendicular *CQ* at a lesser angle *ECQ* than the angle *ACP*; and the line *CE* will always be so situated, that when any circle, described about the center *C*, cuts the line *CA* in *A* and *CE* in *E*, the perpendiculars *AD* and *EF*, drawn from *A* and *E* to the line *PQ*, shall always bear the same proportion to each other; whatever be the magnitude of the angle *ACP*. In water the line *EF* is always three quarters of *AD*.

Angles and sines of incidence and refraction what. Fig. 1, 2.

4. In both these cases the line *AC* is called the Incident Ray, *CB* the Reflected Ray, *CE* the Refracted Ray, *C* the point of incidence, *PCQ* the perpendicular (at the point) of incidence, the angle *ACP* the Angle of Incidence, *BCP* the Angle of Reflection, *ECQ* the Angle of Refraction; the line *AD* the Sine of Incidence, that is, of the angle of incidence; and *EF* the Sine of Refraction, that is, of the angle of refraction.

A Medium what.

5. Empty space, or any transparent body, is called a Medium; and mediums are denser in proportion as they are heavier bulk for bulk; and their power to reflect and refract light is found to be greater in proportion as they are denser, very nearly *.

Laws of reflection and refraction what.

6. The foregoing properties of Reflection and Refraction being discovered and established by repeated experiments upon light and bodies

* Newt. Opt. p. 245. 83.

of

of all sorts both fluid and solid, without any exception yet known; and being the principal foundation of the whole science of Opticks, are called the Laws of Reflection and Refraction; and are expressed by Sir *Isaac Newton* in the following words.

7. *The angles of reflection and refraction lye in one and the same plane with the angle of incidence*; that is, in the plane drawn through the incident ray and the perpendicular at the point of incidence, as represented in the figures, 1, 2. First law.

8. *The angle of reflection is equal to the angle of incidence.* Second law.

9. Hence it follows that the incident and reflected rays are equally inclined to the reflecting plane; that is, the angles ACR and BCS are equal; as appears by taking the equal angles PCA and PCB from the equal angles PCR and PCS. First consequence. Fig. 1.

10. It follows also that when the incident ray is perpendicular to the reflecting surface, it shall be reflected directly back along the same perpendicular; as appears by diminishing the equal angles of incidence and reflection till the rays AC, CB coincide with the perpendicular CP. Second consequence.

11. *If the reflected or refracted ray be returned directly back to the point of incidence, it shall be reflected or refracted into the same line before described by the incident ray.* Third law.

12. *Refraction out of a rarer medium into a denser* [a] *is made towards the perpendicular; that is, so that the angle of refraction be less than the angle of incidence.* Fourth law. [a] Art. 5.

13. *The sine of incidence*, AD, *is to the sine of refraction*, EF, *either accurately or very nearly in a given ratio*; that is, supposing any other incident ray aC to be refracted into the line Ce, and the sines ad and ef to be drawn perpendicular to PQ, the ratio of ad to ef is the same as the ratio of AD to EF. It is found by experience, that if the refraction be made out of air into water, the sine of incidence of red light is to the sine of its refraction as 4 to 3: if out of air into glass as 17 to 11, or nearly as 3 to 2. In light of other colours the sines have other proportions, but the difference is so little that it seldom need be considered. Fifth law. Fig 4.

14. Hence it appears by inspection of the figures (2, 3, 4.) that when the angle of incidence ACP is increased, the corresponding angle of refraction ECQ will also be increased; because the ratio of their sines, AD, EF, cannot continue the same unless they be both increased. Consequently if two angles of incidence be equal to each other, the angles of refraction will also be equal to each other. On the contrary, when the angle of incidence is diminished, the angle of refraction will also be diminished; insomuch that when one of these angles becomes infinitely small the other also becomes infinitely small. First consequence.

 15. And

Second consequence.

15. And so it comes to pass that when the incident ray coincides with the perpendicular to the refracting surface, it will proceed straight forward into the other medium without any bending at all.

Third consequence.

Fig. 2.

16. From which it is reasonable enough to conclude back again, that while the angle of incidence is continually increasing, the refracted ray will be continually more and more bent and diverted from the course of the incident ray produced: I mean if *AC* be continued to *G*, the arch *EG* and the angle *ECG* will continually increase: especially considering that when the angle of incidence in air becomes very nearly a right one, and consequently the incident ray goes almost parallel to the surface of the water, this ray is as much bent at *C* into the line *CE* as the 3[d] figure represents. In which *EF*, the sine of refraction, being always three quarters [a] of *AD*, is now three quarters of the radius of the circle. Hence we find * that this angle of refraction, *ECQ*, is about $48\frac{1}{2}$ degrees: and so the angle *ECS* (being its complement to 90 degrees) is about $41\frac{1}{2}$ degrees; which in this case measures the deviation of the ray from its first course along the surface of the water. The deviation at the surface of glass is greater than at the surface of water; the ratio of the sines being greater, that is, as 3 to 2 or nearer as 31 to 20. Hence we find that the angle *ECQ* is about 40 and *ECS* about 50 degrees.

Fig. 3.

a Art. 13.

Refraction changed into reflection.

17. The bending and deviation is the same when the ray goes back again along the same lines *EC*, *CA*; and if an angle of incidence *eCQ* be any thing greater than about $48\frac{1}{2}$ degrees in water, or any thing greater than about 40 in glass, this ray *eC* will not be refracted into air, but will be reflected into the line *Cf*, making the angle of reflection *QCf* equal to the angle of incidence *QCe*: as will appear by experiment in the sixth chapter.

Experimental proof of these Laws of reflection and refraction.

Fig. 4.

18. The truth of these laws and of all the consequences drawn from them may be easily examined in the manner following. Upon a smooth board *KLMN*, about a center *C* with any opening of the compasses (the larger the better) describe a circle *PRQS*; and having drawn two diameters *PQ* and *RS* perpendicular to each other, from the point *P*, with any opening of the compasses, cut off equal arches *PA*, *PB*, and draw the lines *CA*, *CB*; then sticking three pins perpendicular to the board at the points *A*, *B*, *C*, dip the board into water as far as the line *RS*; and holding it perpendicular to the surface of the water, look along the pins *A*, *C*; and an image of the pin *B* will appear in the water in the line *AC* produced. Which shews that the ray which came from the pin *B* is reflected from the water, at the point *C*, along the line *CA* to the eye of the spectator. If the pin at *C* touches the water, it will disturb the smoothness of its surface;

* By a Table of sines.

and

and therefore it is better not to place it in the center, but a little higher in the line *CA*. The event will be the same if the reflection be made by any other fluid or solid body, as may be tryed by cutting off the lower semicircle, and by placing the diameter, *RS*, of the upper semicircle upon the surface of the solid.

Upon the same board draw the line *AB* cutting *CP* in *D*, and from the lines *DB* and *CS* cut off *DH* and *CI*, each equal to three quarters of *DA*, and through the points *H*, *I*, draw the line *HIE*, cutting the circumference in *E*; and the perpendicular *EF* drawn from *E* upon *PQ*, will be equal to *DH*, or three quarters of *DA*. Then stick another pin at *E*, and the board being dipped into water, as before, the pin at *E* will appear to the eye to be in the same line with the pins at *A* and *C*. Which shews that the ray which comes from the pin *E* is so refracted at *C*, as to advance to the eye along the line *CA*; and therefore when the refraction is made out of water into air, *EF* the sine of incidence, is to *AD* the sine of refraction, as 3 to 4. If other pins be fixed any where in the line *CE*, they will all appear in the line *AC* produced: and the whole line *CE* will appear in the water as if it were a continuation of *AC* straight forward. Which shews that the ray which comes from the pin *E*, describes a straight line in the water; and that it is bent at the surface only. On the contrary, if an opportunity be taken when the Sun is just so high, that the shadow of the pin *A* shall coincide with the line *AC*, the refracted shadow will coincide with the line *CE*. Or whatever be the Sun's height, move the pin *A* higher or lower till the shadow falls upon the center *C*, and there fix it, suppose at *a*; then sticking the compasses into any point of the refracted shadow, take up the board, and through this point and the center *C* draw a line *Ce*, cutting the circle in a new point *e*; and the ratio of the new perpendiculars, *ad* and *ef*, will be the same as before; that is, as 4 to 3, as near as can be measured.

19. Lastly it is to be observed, that a ray of light is reflected or refracted at a spherical surface according to the same laws as if it were reflected or refracted at a plane, touching the spherical surface at the point of incidence. Let *AC* be a ray of light falling upon any point *C* of a spherical surface *MCN*, represented by the arch *MCN*, whose center is *O*; through the points *O* and *C* draw the line *PQ*, and the line *RCS* perpendicular to it, representing a plane surface touching the spherical surface at *C*. Now because a ray of light is considered as a physical line, and is refracted or reflected at a physical point [a], which is common to both surfaces, *MCN* and *RCS*, it follows that the refracted or reflected ray will take the same course in both cases. And this argument is also confirmed by universal experience.

This proof applied to spherical surfaces.

Fig. 5, 6.

[a] Art. 18.

CHAP.

CHAP. II.

CONCERNING GLASSES.

Design.

HAVING considered the reflection and refraction of a single ray at a single surface, I proceed to shew how one or more pencils or parcels of rays are reflected and refracted, first by a single surface alone, and then by several surfaces of plane and spherical glasses; and also how images or pictures of objects are formed by them. Though the manner and reason of all these effects of glasses upon light, may be sufficiently explained by the laws or experiments above mentioned[a] without geometrical propositions; yet, I think, the exact quantities of these effects cannot be determined by so few experiments, without the assistance of geometry. Therefore to avoid the use of it in this first book, I have added a few more experiments to the end of this chapter, by which those quantities may be easily determined.

a Art. 6.

An object what and how it radiates.

20. As rays of light are incessantly thrown out and dispersed in all possible directions from every point of a luminous body; so when they illuminate other bodies, on which they fall, they are also incessantly thrown back from every point of these bodies. For the points of opake bodies so enlightened are visible to the eye at any point of space and in any point of time, as well as the points of the luminous body that enlightened them. The numberless rays which flow from all visible bodies, called objects, may be methodically distributed in this manner. The surface of the object is considered as consisting of physical lines, and these lines as consisting of physical points, and these points are conceived to radiate all manner of ways. It is usual to make use of nothing else for an object but a physical line. For by how much that line is increased or diminished in apparent magnitude or brightness or distinctness, so much the diameter or length of any object, in its place, would be increased or diminished.

A focus, pencil, parallel rays what Fig. 9.

21. The point Q from which rays diverge, or towards which they converge (being made to go back towards the same point though they may never meet at it) is called their focus. And in both cases any parcel of these rays, as QBC or QBA, considered apart from the rest, is called a pencil of rays; and these rays are said to belong to that focus, whether it be near at hand or at an immense distance; and in the latter case the rays are called, and considered as, parallel or equidistant from each other; because the difference of their distances at any two given places is insensible.

Reflection of a pencil of parallel rays at a plane surface.

22. The 7th and 8th figures represent a pencil of rays, QC, which falling in parallel lines upon a plane polished surface, represented by the line ACB, are reflected from it into as many other parallel lines, Cq;

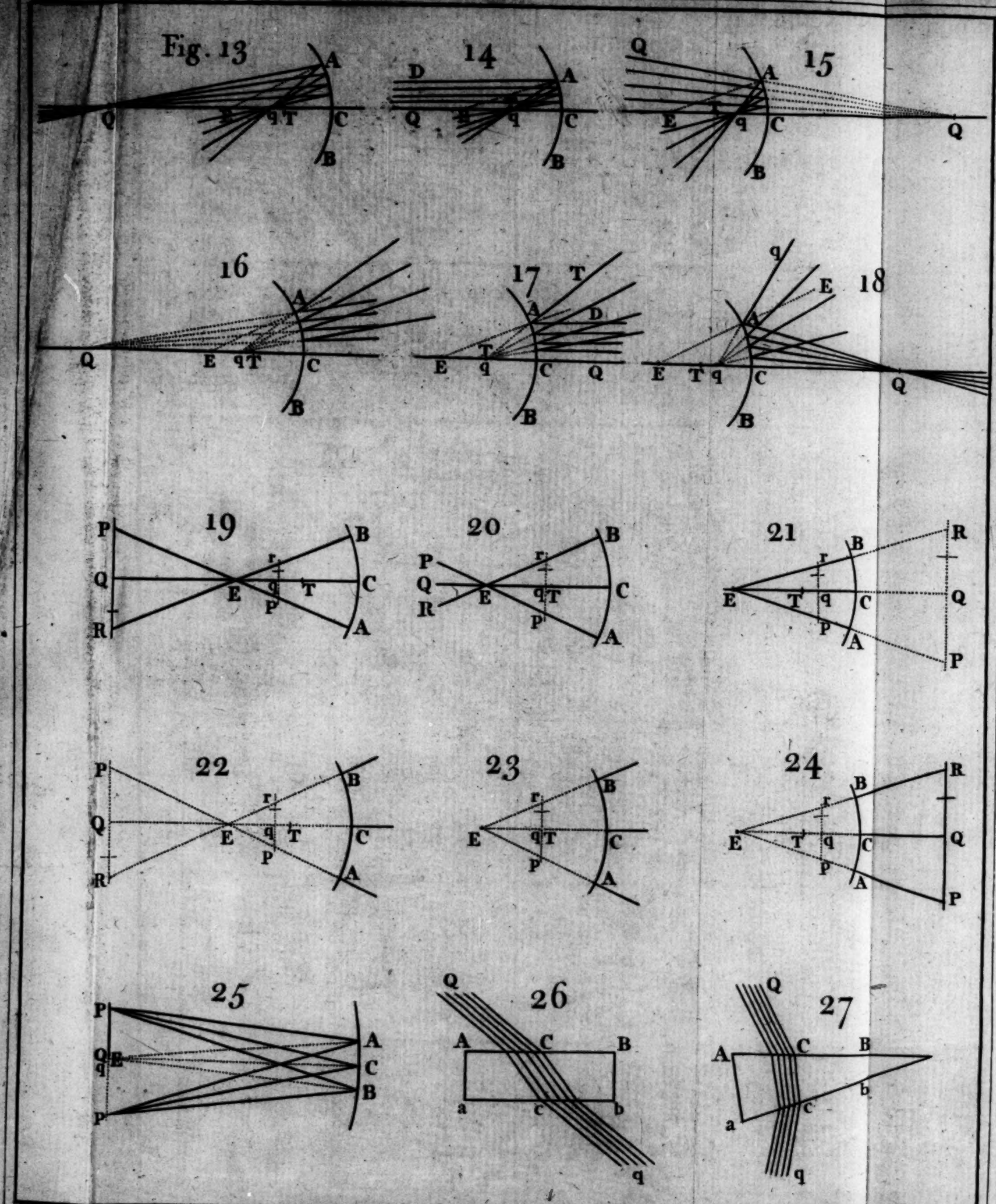
Fig. 13
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Cq; which are inclined to that plane just as much as the incident rays were inclined to it [a].

[a] Art. 9.

23. The 9th figure represents the manner in which the rays of a pencil QAB, diverging from any point of an object Q, and falling upon a straight line ACB, or upon a polished plane represented by it, do all diverge after reflection as if they came from another point q. The ray QC, which falls perpendicularly upon the plane AB, is reflected back again along the same [b] line CQ; but all the rest falling upon it with greater and greater degrees of obliquity as the points of incidence lye farther and farther from C, are also reflected with degrees of obliquity respectively greater [c]. It will seem reasonable therefore, especially by attending to the figure, that the reflected rays, produced backwards, should meet the perpendicular QC, produced, in a point q, situated as far from the reflecting plane on one side, as Q is on the other: and consequently that all the rays flowing from a single point Q, will after reflection diverge from a single point q at an equal distance on the other side of the reflecting plane.

And of diverging rays.

[b] Art. 10.

[c] Art. 9.

24. On the contrary if q be a focus to which the incident rays are made to converge, by ways hereafter described, the point Q will be their focus after reflection [d] from the surface ACB.

And of converging rays.

[d] Art. 11.

25. What has been said of the point Q is applicable to every other point of an object PQR: namely that as the focuses Q, q lye at equal distances on each side of the reflecting plane, so the focuses P, p lye on each side at other equal distances, and R, r at other equal distances, in lines Pp, Rr, drawn perpendicularly through the plane AB. Hence it is easy to understand by inspection of the figures, that these focuses p, q, r, with innumerable others, lying all in the same order as the corresponding points P, Q, R, compose an imaginary line of the same length and shape as the line PQR; and that the situation of the line pqr, with respect to the backside of the reflecting plane, is the very same as that of PQR with respect to the foreside of it. This line pqr is called an image or picture of the object PQR.

And of several pencils which form images. FIG. 10, 11. 12.

26. The 14th and 17th figures shew that parallel rays falling upon an arch of a reflecting circle ACB, or upon a concave or convex surface represented by it, are so reflected as to converge to a focus T, when they fall upon the concave side of the surface, and to diverge from T when they fall on the convex side. In both cases the ray QC, which passes through, E, the center of the surface, and falls perpendicularly upon it at C, is reflected back again along the same [e] line CQ; but all other rays being parallel to QC fall upon the surface with various degrees of obliquity, by reason of its continued curvity. Every ray as it is more remote from QC, makes a greater angle of incidence DAE, with the perpendicular EA at the point of incidence; and consequently the

Reflection of a pencil of parallel rays at a spherical surface. Fig. 14, 17.

[e] Art. 10, 19.

[a] Art. 8, 19. the equal [a] angle of reflection EAT becomes greater continually as the point of incidence, A, is farther from C. It is reasonable therefore that all the reflected rays should converge and convene pretty close together about some certain point T of the direct ray QC, if the reflecting surface be concave, or else diverge from it, if the surface be convex. By reasoning farther, as well as by experience, it is found that in both cases the focus T divides the semidiameter CE into two equal parts.

Reflection of diverging and converging rays. Fig. 13. to 18. [b] Art. 11.

27. In the foregoing cases, if the said point T be a focus of incident rays, the reflected rays will all be parallel to the line CTE drawn through the center E [b]. But if the focus T be removed to any point q towards E, the angles of incidence, as qAE, and consequently the equal angles of reflection, as EAQ, will all be diminished; and if q be removed towards C, they will all be increased. Consequently the reflected rays, as AQ, which before were parallel to the direct ray EC, will now be inclined to it; so as to belong to another focus Q situated on the same side of T as q. From the contemporary decrease of the angles of incidence and reflection while q is moved from T towards E, and their contemporary increase while it is moved towards C, it follows that the focuses q, Q move contrary ways, so as to meet each other at the center E, or at the surface C, if the arch AC be very small. It is observable that the properties of concave and convex surfaces are just alike, and are changed into one another by conceiving the incident rays to come the contrary way in the same lines produced.

Reflection of several pencils which form images. Fig. 19. to 24.

28. These figures shew the manner in which the image or picture pqr, of an object PQR, is formed by rays reflected from a concave or convex surface ACB. As the focus q has been shewn to lye in the perpendicular ray QC, drawn from Q through the center E, just so the focus p, of the pencil of rays that flow from another point P, will be in the perpendicular ray PA, drawn through the center E. For every ray that passes through the center or tends towards it, falls perpendicularly upon the arch or surface ACB, and all others fall obliquely upon it.

Some general properties of images.

29. Hence it is easy to understand that if the object PQR be so small, or so far distant from the reflecting surface or its center E, that all the points P, Q, R be nearly at equal distances from it, then all the points p, q, r, of the image, will be nearly at other equal distances from the said surface or its center. It is also to be observed, that when the object and its image lye both on the same side of the center, the image is upright, but when they lye on contrary sides of the center, it is inverted; and that it is greater or less than the object in proportion as it lies at a greater or less distance from the center than the object it self. These things are plain by inspection of the figures, observing that the object and image are both terminated by two lines Pp, Rr, that meet or cross in the center E. Hence it follows that the image will be nearly equal

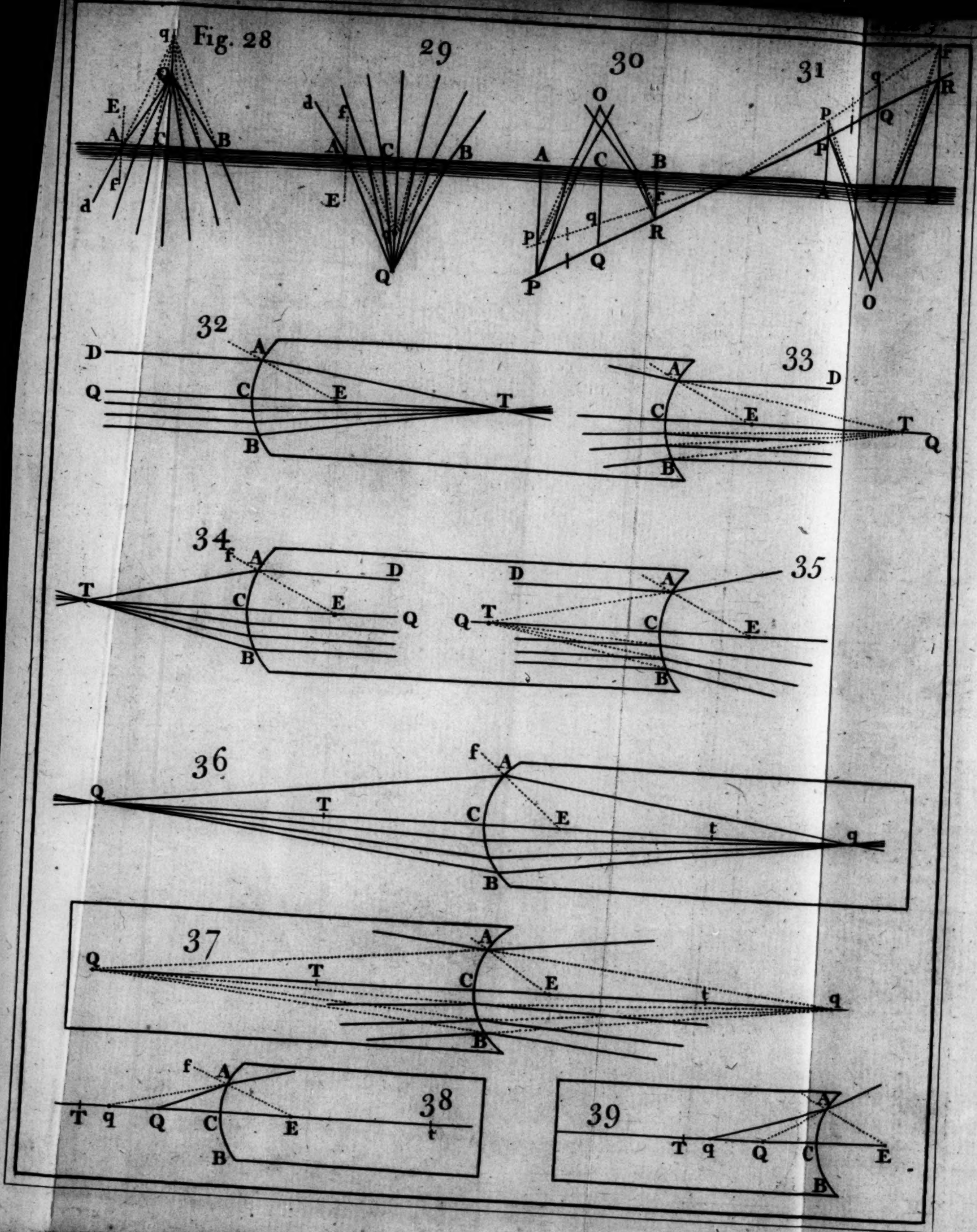
Fig. 28
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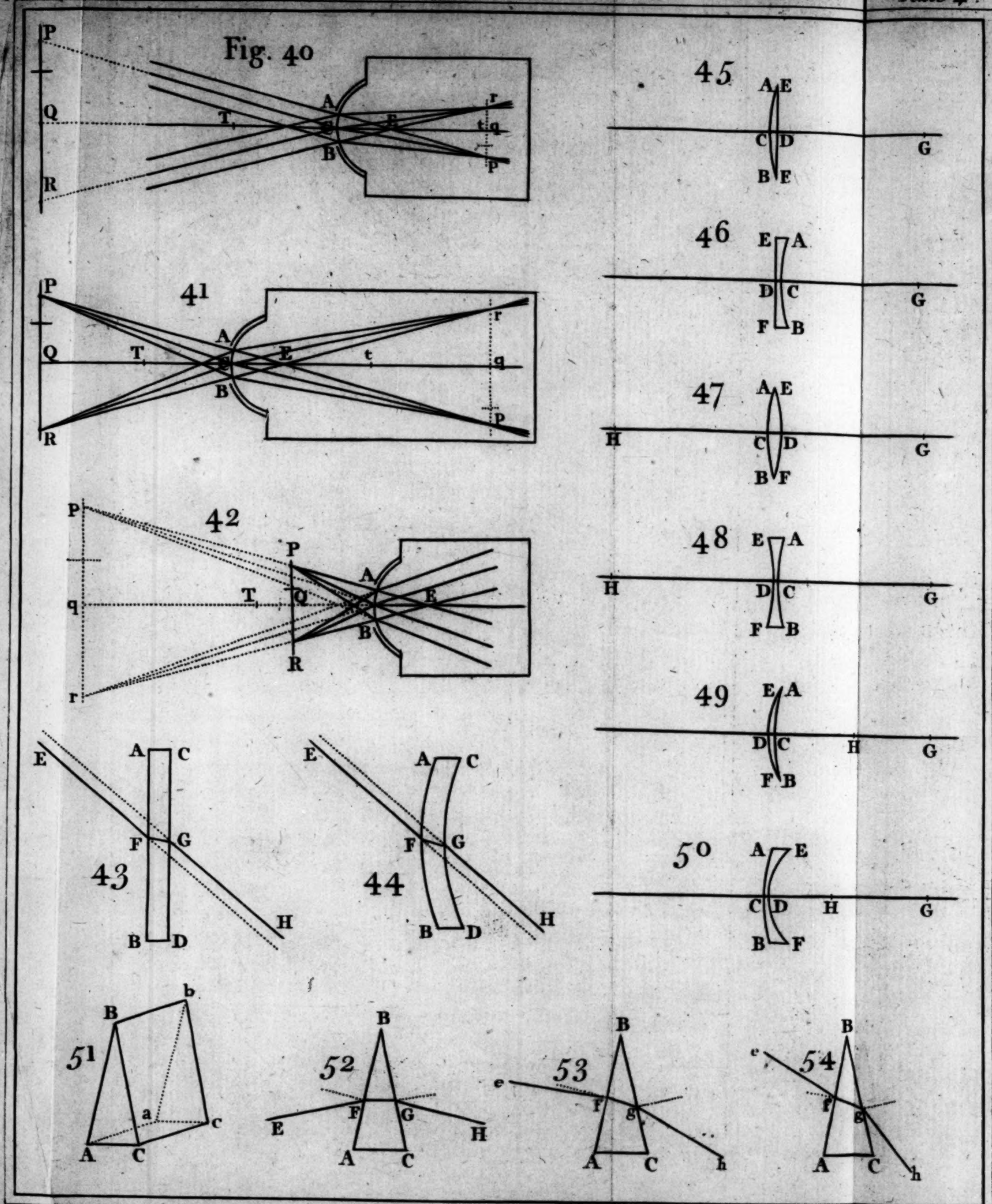
Fig. 40
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equal to the object when they meet [a] each other at the surface, and also when they meet at the center. For in this latter case the rays which flow from Q placed in the center, will be reflected directly back to q in the same center; and taking Ep equal to EP, since EC is perpendicular to them both, the angles PCE, ECp will be equal; and so the ray PC will be reflected to p: and when any other point of incidence as A is not far from C, the line AE will also be nearly perpendicular to Pp, and so the angles PAE, EAp being nearly equal, the ray PA will be reflected nearly to the same point p as the ray PC was reflected to. I proceed now to refractions.

a Art. 27. Fig. 25.

Refraction of a pencil of parallel rays at a plane surface.

30. In the 26th figure QC represents a pencil of parallel rays falling obliquely upon a straight line ACB, or upon a plane surface represented by it, which after refraction are also parallel among themselves; every one being equally bent. Because when the angles of incidence are all equal among themselves the angles of refraction are also equal among themselves [b]. For the same reason if these rays be refracted again at another plane, either parallel or oblique to the former, they will still be parallel among themselves after every refraction. In strictness this is only to be understood of rays of the same colour: as will be explained in the 6th chapter.

b Art. 14. Fig. 26, 27.

Refraction of a pencil of diverging and converging rays. Fig. 28, 29.

31. The rays of a pencil QAB, diverging from Q, and falling upon a straight line ACB, or upon a plane surface represented by it, are so refracted as to diverge from another point q situated in that ray QC (produced) which falls perpendicularly upon the plane. For this ray passes straight through the surface [c], but all the rest, as QA, are bent; and every one so much the more as the point of incidence A is remoter from C [d]; because the angle of incidence QAE, and consequently that of refraction grows larger [e]. For this reason all the refracted rays will diverge pretty nearly from a certain point q on the same side [f] of the surface AB as Q. It is found by other arguments and also by experience, that if the refracting body be glass, the greater of the two focal distances QC, qC, is to the lesser as 3 to 2; and if it be water, as 4 to 3; that is, in the proportion of the sines of incidence and refraction in those several mediums [g]. On the contrary if the incident rays be made to converge towards q, the refracted rays will converge to Q [h].

c Art. 15. d Art. 16. e Art. 14. f Art. 11, 12. g Art. 13. h Art. 11.

Refraction of several pencils which form images.

32. The 30th and 31st figures represent an image pqr of an object PQR, formed by a refracting plane ACB, in the manner described in the 25th article. The ratios of Ap to AP, Br to BR, &c. are all equal.

Refraction of a pencil of parallel rays at a spherical surface.

33. The 32d, 33d, 34th and 35th figures shew in what manner a pencil of parallel rays falling upon an arch of a circle ACB, or upon a spherical surface represented by it, do after refraction converge to or diverge from a focus T. The ray QC which passes through E, the center of the surface, and consequently falls perpendicularly upon it,

goes straight through it without refraction [a]. But all the rest, being parallel to *QC*, fall with various degrees of obliquity upon the surface by reason of its continued curvity; and every one of them as it is remoter from *QC* falls more and more obliquely and consequently is more and more bent [b]. It is reasonable therefore that the refracted rays should converge and convene pretty close together about some certain point *T*, in the unrefracted ray *QC* (produced) if they bend towards that ray; or else diverge from *T*, if they bend from it. Which way they are bent will appear by drawing a perpendicular *EA* to the surface at *A*, and by considering the position of the denser medium [c]. Hence it will appear that if the surface of the denser medium be convex, the refracted rays will converge to *T*; and diverge from *T*, if it be concave. It is found by farther reasoning, and also by experience, that if the refracting body be glass, the greater of the two distances *CT*, *TE* is to the lesser as 3 to 2; and as 4 to 3 if it be water or ice; that is, in the proportion of the sines that determine the refractions in those bodies.

a Art. 15.
b Art. 16.
c Art. 12.

Refraction of a pencil of diverging and converging rays at a spherical surface. Fig. 36 to 39.

34. In the foregoing cases if *T* be a focus of incident rays the refracted rays will be parallel to the perpendicular ray *TC*, that passes through *E* [d]. But if this focus *T* be removed to any other point *Q* in the line *TC* produced, the angles of incidence and refraction will either be increased all together, or diminished all together; and consequently all the refracted rays, which before were parallel to *TC*, will now be inclined to it; so as to belong to another focus *q*, on the contrary side of the surface to *Q*, if *Q* be farther from the surface than *T*; otherwise on the same side of the surface. One of the chief properties of these corresponding focuses *Q*, *q*, is this that follows. Since the angles of incidence and refraction do both increase or else both decrease together [e], it follows that the focuses *Q*, *q* must both move the same way in the line *QE* produced. And consequently, while both lye on the same side of the surface or of its center, they must both move from it or both towards it; and if they move towards it, they will come nearer together, till they both coincide at the center or both at the surface, when the arch *AC* is very small. But if the focuses *Q*, *q* be on contrary sides of the surface or of its center; while one moves from it, the other will move towards it; and on the contrary.

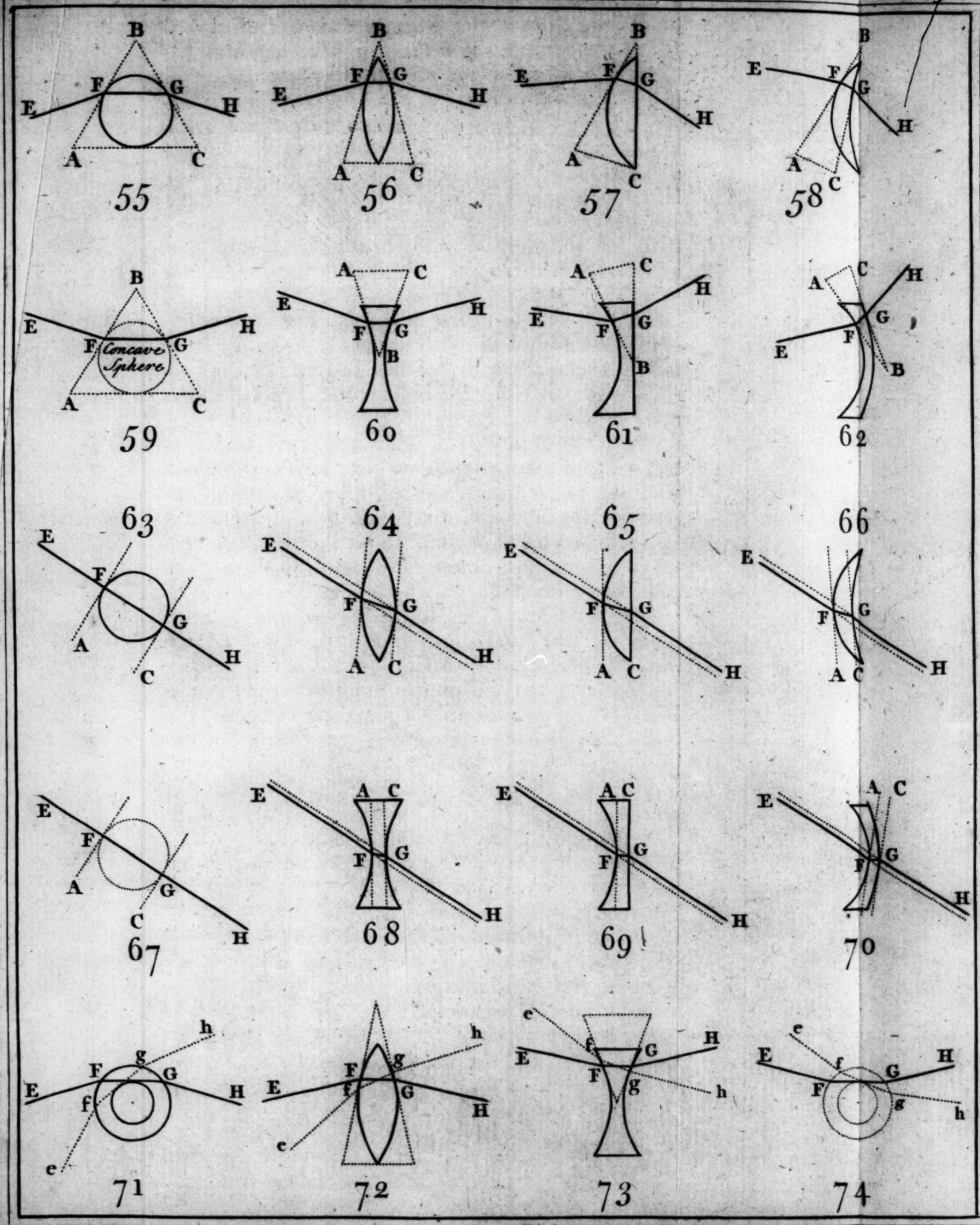
d Art. 11.
e Art. 14.
Fig. 38, 39.
Fig. 36, 37.

Refraction of several pencils which form images at a spherical surface.

35. The 40th, 41st and 42d figures shew in what manner the image *pqr* of an object *PQR* is formed by several pencils of rays, (refracted by a spherical surface,) whose axes or unrefracted rays are *PEp*, *QEq*, *REr*. The properties of these images are the same as of those made by reflection from a spherical surface, and are already described in the 29th article.

Refraction of a ray through parallel plane surfaces.

36. A ray of light *EF* falling obliquely upon a flat piece of glass, or any medium terminated by two parallel planes represented by the lines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lines AB, CD, will emerge from it after both refractions at F and G in a line GH parallel to the incident ray EF. For since any line FG which the ray describes in passing between the parallel planes, is equally inclined to them both, it will be bent as much at G in going forward, as it would be at F in going backward [a]; and these equal bendings being made contrary ways, the incident and emergent rays EF and GH are therefore parallel. Fig. 43.

a Art. 11.

37. The lines described by the incident and emergent rays EF and GH, being produced are closer together when the glass is thinner, and also when the ray falls less obliquely upon it; because the bendings at F and G are then less [b]: and in these cases if the glass be not flat but bent a little as represented in the 44th figure by two parallel arches AB, CD, the lines EF, GH will still be nearly parallel. For the bended surfaces refract the ray $EFGH$ just as much as two planes would do supposing they touched the surfaces at F and G [c]: and these planes will be nearly parallel when the line FG is but little inclined to the surfaces; being exactly so when it stands perpendicular to them both.

Refraction of a ray through parallel spherical surfaces.

b Art. 16.

c Art. 19.

38. A thin piece of glass or of any transparent substance bounded on one side by a polished plane surface, represented by the line EF, and on the other side by a small portion of a polished spherical surface, represented by the arch ACB; or bounded on both sides by spherical surfaces ACB, EDF, is called a lens or simply a glass; and is conceived by mathematicians to be generated or described by turning the figure $ACBFDE$ round about the line CD, drawn through the middle of it perpendicularly to both its sides. This line CD produced is therefore called the axis of the lens; and it passes through G and H, the centers of its surfaces. The points C, D where it cuts the surfaces are called the vertexes of the lens, and the middle point between them is called its center. The 45th figure represents a plano-convex glass, the 46th a plano-concave, the 47th a double-convex, the 48th a double-concave, and the 49th and 50th two concavo-convex glasses, whereof the first is called a meniscus, because it resembles a little moon. It must be remembered once for all, that the thickness CD of all these glasses is generally so small, that it seldom need be considered.

A lens what. Fig. 45 to 50.

39. A glass prism is a body, shaped like a wedge, that has three edges, being bounded with two equal and parallel triangular ends ABC and abc, and three plane and well polished sides, which meet in three parallel lines Aa, Bb, Cc, running from the three angles of one end to the three angles of the other: and when it is viewed endways it is represented only by a triangle ABC, as in the 52d figure.

A prism what. Fig. 51.

40. When a ray of light $EFGH$ is refracted at F and G in passing through the sides, AB, BC, of a prism, the course of the emergent ray, GH, always deviates from, EF, the course of the incident ray, towards

Refractions of a single ray through a prism. Fig. 52, 53, 54.

wards the thicker part of the prism, more or less ,as the refracting angle *ABC* is greater or smaller. And if the refracting angle be given (or invariable) and the refractions be but small, the quantity of deviation will also be given, though the position of the incident ray be varied at pleasure.

Fig. 52. For supposing at first that the ray *FG*, within the prism, is equally inclined to its sides *AB*, *BC*, as in fig. 52, it is evident from the position of the perpendiculars to those sides at the points *F* and *G*, that both the refractions are made from the edge *B* towards the opposite side *AC*[a].

a Art. 12.

Fig. 53. Now let *FG* become unequally inclined to the sides *AB*, *BC*, by turning it gradually into the position *fg*; and while it becomes less and less oblique to one side, suppose *AB*, it will become more and more oblique to the other side *BC*. Consequently supposing a ray to go both ways along this variable line *fg*, it will be more and more bent in going through the side *BC* and less and less in going back through the side *AB*; so that the total bending of the ray, consisting of both its bendings, or angles *efg* and *fgh*, taken together, will continue to be much the same in all its positions. The circulation of the line *fg*, may be farther continued till it becomes perpendicular to the side *AB*; and then the bending at this side is nothing: it may also be continued still farther till the bending at *f* is made the contrary way; which still takes off from the perpetual increase of the greater bending at *g* and keeps the total bending invariable.

Fig. 54.

Fig. 53. When *fg* is perpendicular to *AB*, let the latter plane *BC* be turned gradually towards the former *BA*, upon the edge *B*, and the ray that comes along *fg* will gradually fall less obliquely upon it; and consequently the bending at *g* will be gradually diminished[b]; and reduced to nothing when the refracting angle *ABC* vanishes. Lastly if several rays be supposed to come parallel to one another they will all emerge parallel to one another[c]. Therefore the quantity of deviation of a ray does not at all depend upon its passage through a thicker or thinner part of the prism, nor upon its inclinations to the sides of the prism, but is proportioned to the quantity of the refracting angle *ABC*; and the more exactly as this angle and the refractions at its sides are smaller.

b Art. 14, 15.

c Art. 30.

Refractions of a single ray through the edge of a lens. Fig. 55 to 62.

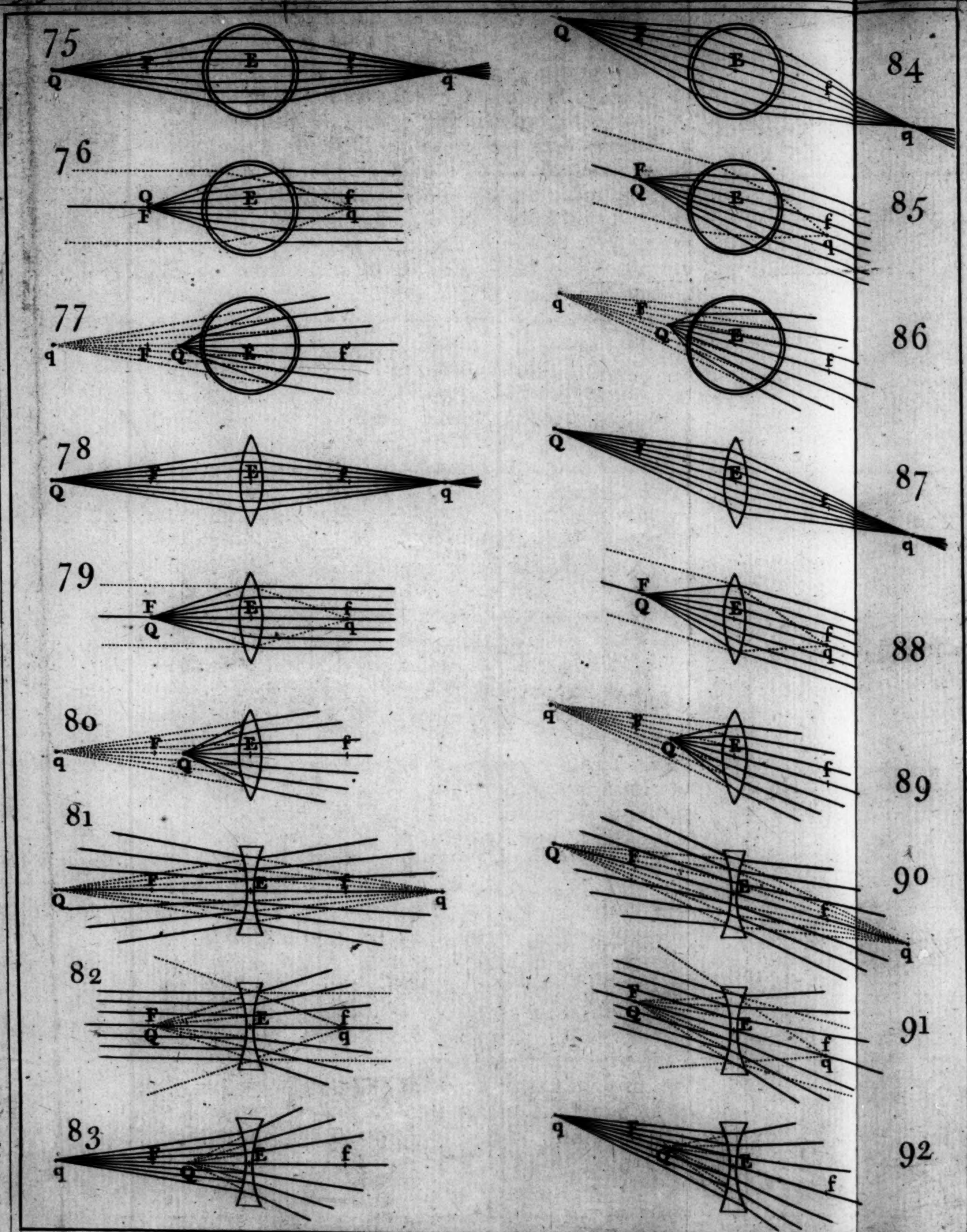
41. For the same reason when a ray of light *EFGH* passes through the edge of a convex or concave lens, or the sides of a globe, its emergent part *GH* always deviates from the course of the incident part *EF* towards the thicker part of the glass. Because the refractions at *F* and *G* are the same as if they were made by two planes *FA*, *GC* that touch the spherical surface at *F* and *G*[d]; and so the sides of the glass may be considered as inclined to each other like the sides of a prism.

d Art. 19.

Refractions of a single ray through the middle of a lens.

42. It follows therefore from the two last articles, that the deviation of the course of the emergent ray, from that of the incident ray, is

gradually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gradually diminished as the ray goes nearer and nearer to the middle of the glass: till, when it goes through the middle, its emergent and incident parts are either parallel to each other, or else are one continued line, when the ray coincides with the axis of the glass. For the angle made by the touching planes, *FA*, *GC*, is gradually diminished as the ray *FG* approaches to the middle: till at last it vanishes when they become parallel, as in the 36th article. Fig. 63 to 70.

This ray is considered as straight, and is called the axis of a pencil.

43. When a pencil of rays falls upon any glass, that ray which passes through its center, or middle point, is called the axis of the pencil. And because its incident and emergent parts, *EF* and *GH*, are either one continued line or two parallel lines[a], its whole course in optical experiments may be always taken for one straight, physical line: from which it differs insensibly when the thickness of the glass is small, and when the pencil falls not too obliquely upon it. Because the parallel lines *EF* and *GH* produced, go closer together in proportion as the line *FG* is shorter, and as the bendings at *F* and *G* are smaller.

[a] Art. 42.

Rays are equally bent which pass at equal distances from the center of a globe. Fig. 71, 74.

44. All rays, as *EFGH* and *efgh*, which cross each other in a refracting globe and pass through it at equal distances from its center, so as to touch a concentrick globe, are equally bent. For in this case the chords *FG*, *fg* being equal, their obliquities to the surface of the globe are also equal, and consequently the bendings of the ray *EFGH* at *F* and *G*, both severally and together, are equal to the bendings of the ray *efgh* at *f* and *g*: as is evident by conceiving the rays to go both ways along the chords *FG*, *fg*. Therefore the angle made by the incident and emergent parts of one ray, produced till they meet, will be equal to the angle made by the incident and emergent parts of the other ray produced till they meet; which is what I mean when I say the rays are equally bent.

And from the center of a lens. Fig. 72, 73.

45. All rays, as *EFGH*, *efgh*, which cross each other at any given point of a lens, or which pass through it at equal distances from its center, are equally bent, provided they do not fall very obliquely upon it. Imagine a line *FG* within the glass, at first to be equally inclined to its sides, and then to be turned a little about any point of it, till it comes into the position *fg*; and while it becomes more and more oblique to one side of the glass, suppose *Ff*, it will become less and less oblique to the other side *Gg*. Consequently if a ray be supposed to go both ways along this variable line *fg* it will be more and more bent in going through the side *Ff* and less and less bent in going through the other side *Gg*[b]: so that the total bending of the ray consisting of both its bendings, or angles *efg*, *fgh*, taken together, will continue to be much the same in all its positions. The circulation of the line *fg*, about the given point, may be farther continued till the bending at *g* is diminished to nothing; and still farther till it be made the contrary way;

[b] Art. 16.

way; (as was explained in the 40th article;) which ſtill takes off from the perpetual increaſe of the greater bending at f and keeps the total invariable. To keep the ſame bending it is only neceſſary that the rays FG, fg ſhould keep at equal diſtance from the axis of the lens as near as poſſible: and nothing alters the total bending but the alteration of that diſtance [a]; becauſe the inclination of the tangent planes, like the refracting angle of a priſm, will then only be altered.

a Art. 40.

Refraction of a ſingle pencil of parallel rays by any glaſs. Fig. 76, 79, 82, 85, 88, 91.

46. When a large pencil of parallel rays falls either directly or a little obliquely upon the whole ſurface of any glaſs, which is thicker in the middle than at the edges; all the emergent rays will be bent from all ſides towards that ray which goes through the middle of the glaſs: and on the contrary, if the glaſs be thicker at the edges than at the middle, they will all be bent outwards from the middle ray [b]. And becauſe in both caſes the bendings are equal at all equal diſtances round about the middle; and grow greater at greater diſtances from the middle [c]; the emergent rays will all converge pretty nearly to ſome certain point F in the middle ray, if the glaſs be convex; or diverge from a certain point F, if the glaſs be concave.

b Art. 41.

c Art. 40.

Refraction of parallel rays coming contrary ways.

47. When parallel rays come contrary ways and fall upon oppoſite ſides of any lens, the diſtances of the focuſes of the emergent rays from each ſide of the center of the lens will be equal; though the ſemidiameters of the ſurfaces of the lens be never ſo unequal, or though one ſide be plane and the other ſpherical. For ſince any two rays which come directly oppoſite to each other, are at equal diſtances from the common axis of the pencils, and after croſſing each other and emerging from the glaſs, are equally bent from their firſt courſe [d], theſe two emergent rays (produced) will meet the axis at equal diſtances EF, Ef from the center of the glaſs. When the rays come parallel to the axis of the lens, their two focuſes F, f are called the principal focuſes of the lens: and EF or Ef is called its focal diſtance, and by ſome authors its focal length.

d Art. 44, 45.

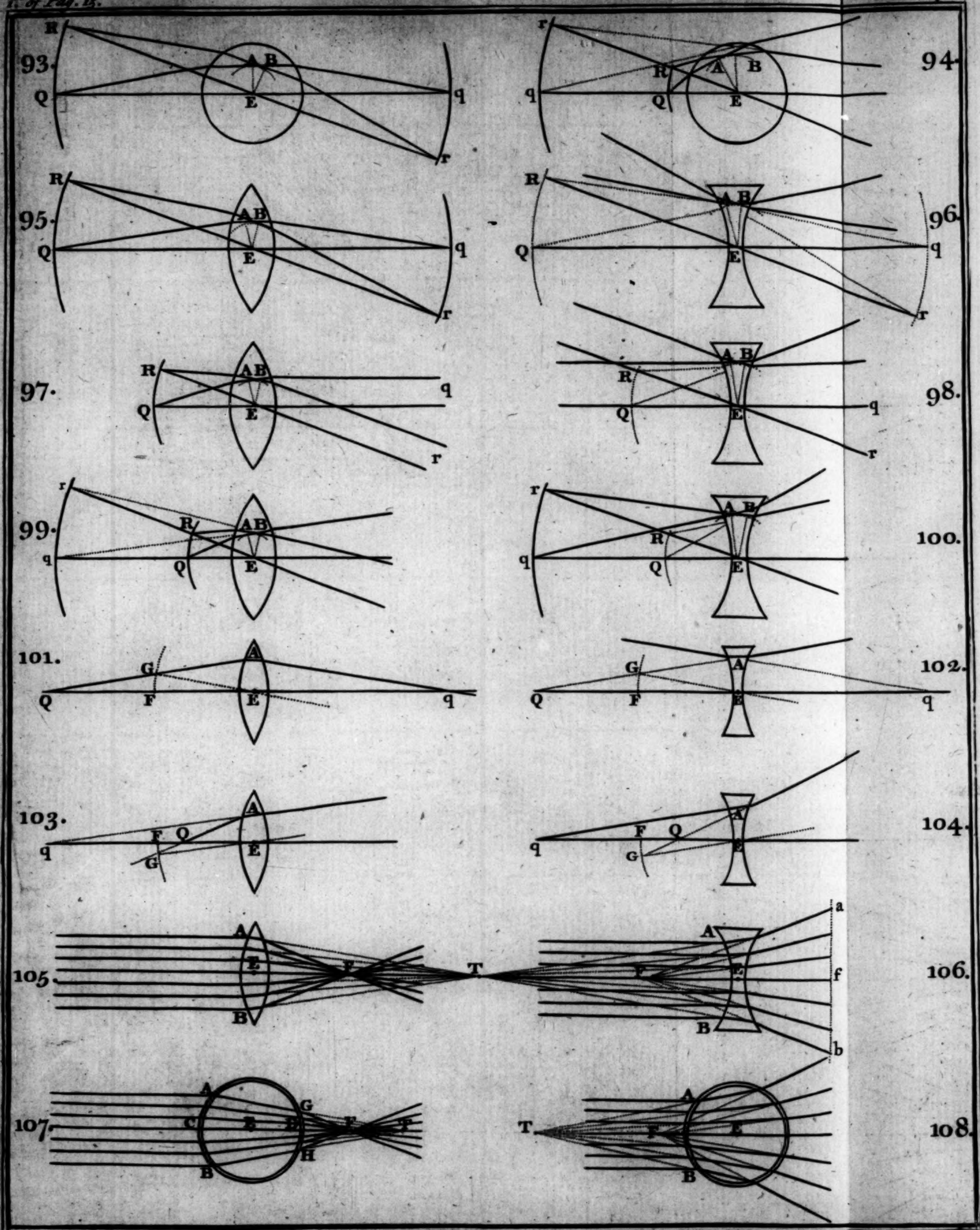
Principal focus, and focal diſtance, what.

Refraction of a ſingle pencil of diverging or converging rays through any glaſs. Fig. 75 to 92.

48. On the contrary if the rays be returned directly back from the focus F, the emergent rays will all be parallel to FE, the axis of the pencil [e]. Conſequently if this focus F be removed to Q, farther from the glaſs, the emergent rays will belong to another focus q on the contrary ſide of the glaſs; but if Q be put nearer to the glaſs than F, the emergent rays will belong to a focus q on the ſame ſide of the glaſs as Q. Becauſe while the rays are put into theſe different ſituations their bendings will not be altered, if they keep their reſpective diſtances from the center of the glaſs [f]. Conſequently if either of the correſponding focuſes Q, q be put in motion along the axis of the pencil, either direct or oblique, the other focus will move the ſame way: and therefore if theſe focuſes be on contrary ſides of the glaſs, while one moves towards

e Art. 11.

f Art. 44, 45.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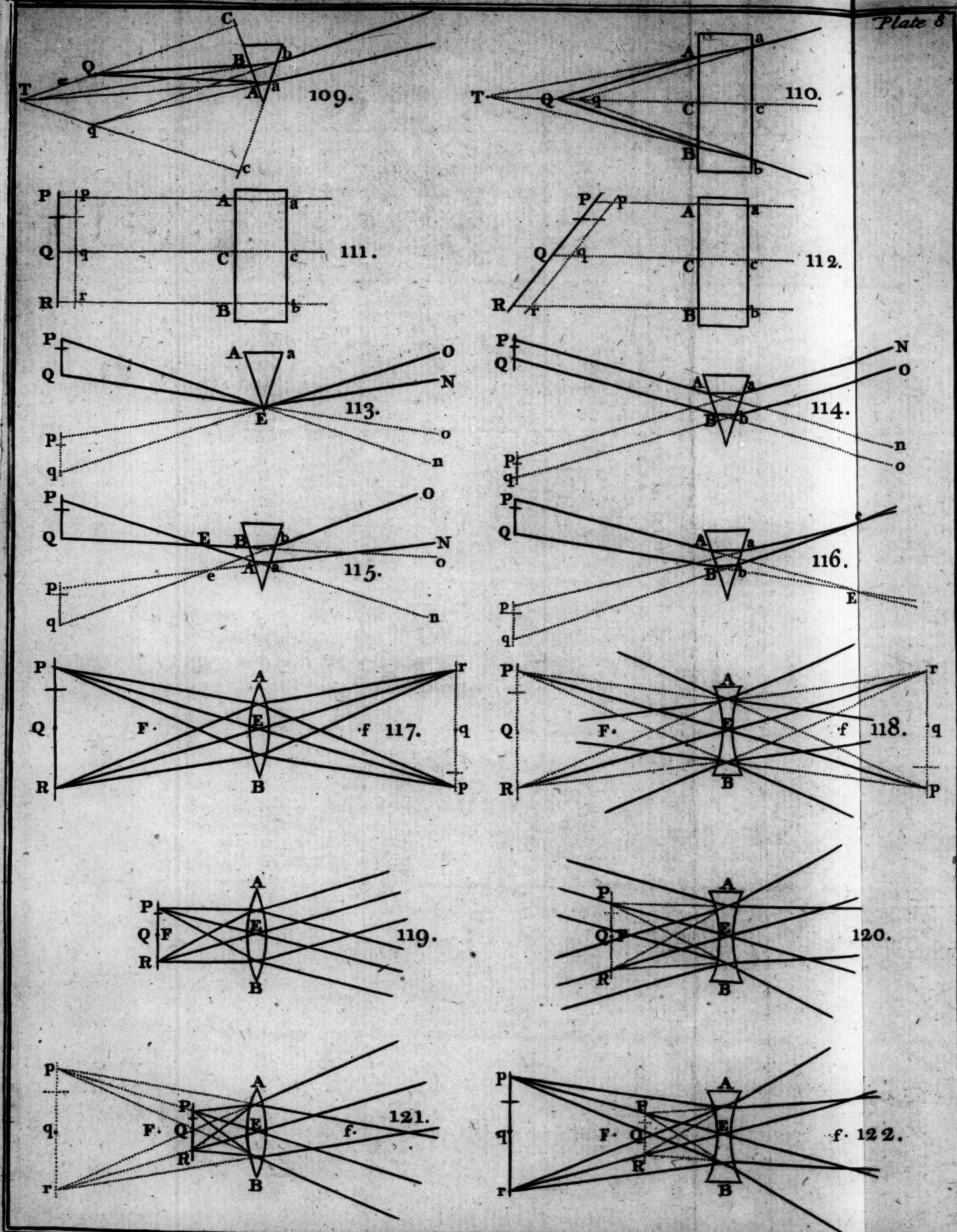
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towards it the other will move from it; but if they be both on the ſame ſide of the glaſs, they will both move from it or both towards it; and will come nearer to each other as they come nearer to the glaſs, till when one coincides with its ſurface the other will do ſo too very nearly, provided the glaſs be very thin and the diſtance of the ray from its axis be very ſmall. Theſe focuſes cannot therefore coincide at the ſurface of a globe, becauſe the points of incidence and emergence are too far aſunder.

It is obſervable that the properties of concave ſurfaces and glaſſes are the ſame as of convex ones; which will appear by conceiving the rays to come contrary ways in the ſame lines produced; and accordingly to be changed from diverging to converging, and on the contrary: as repreſented in the figures, by the black lines and pointed lines.

Refractions of ſeveral pencils of diverging and converging rays. Fig. 93 to 100.

49. If ſeveral focuſes Q, R of incident rays be at any equal diſtances EQ, ER from the center of any glaſs, the focuſes of the emergent rays will alſo be at other equal diſtances Eq, Er from the ſame center in the lines EQ, ER produced; provided none of the rays fall very obliquely upon the glaſs. Take any point A within the glaſs, not far from its axis Qq, through which a ray paſſes from the focus Q to the focus q. Draw the line AE and while the figure $QAEq$ is conceived to turn a little about the center E, into the poſition $RBEr$, the extremities of the lines EQ, EA, Eq will deſcribe ſmall arches QR, AB, qr about that common center E. Then let another ray which belongs to the focus R be refracted through the point B, and after emergence it will belong to the point r; becauſe the total bendings of two rays, QAq, RBr, which paſs at equal diſtances, AE, BE, from the center of the glaſs, are equal[a]. And the reſt of the rays that belonged to R will belong to the ſame point r; becauſe it is ſituated in the axis of the pencil[b].

a Art. 44, 45.
b Art. 46.

Refraction of ſeveral pencils of parallel rays. Fig. 97, 98.

50. Hence the focuſes of all pencils of parallel rays that fall not too obliquely on the ſame or on oppoſite ſides of any glaſs are equidiſtant from its center. For the argument continues the ſame while the equal diſtances Eq, Er, are equally increaſed till they become infinite, that is till the rays of each pencil become parallel.

Having the focus of incident rays upon a lens to find the focus of the emergent rays. Fig. 101 to 104.

51. Hence if Q, the focus of incident rays be given, and q, the focus of the emergent rays be required, draw QE, the axis of the pencil, and with the center E and ſemidiameter EF, equal to the focal diſtance of the lens, (to be found by experiment,) deſcribe an arch FG cutting any incident ray, QA, in G; join EG and drawing Aq parallel to it, the point q where it cuts the axis of the pencil will be the focus of the emergent rays. For ſuppoſing other rays, beſides GA, to flow from or towards G, they will all emerge parallel to their axis GE produced[c].

c Art. 50.

52. The

Refraction of rays through several surfaces otherwise considered. Fig. 105 to 110.

52. The refraction of a pencil of rays through all sorts of glasses may also be considered in this manner. By the refraction at the first surface AB the rays are disposed within the glass to converge to or diverge from a focus T; which may be considered as the focus of incident rays upon the second surface; by whose refraction they are all directed to another focus F. For example, let Q be the focus of incident rays upon a glass prism; QC perpendicular to its first side AB. To QC add QT, equal to half QC; and T will be the focus of the rays QA, QB, &c. after refraction at the surface AB[a]; and being also the focus of incident rays at a and b upon the second surface ab, from Tc drawn perpendicular to ab take away Tq equal to a third part of Tc; and q will be the focus of the emergent rays qa, qb produced.

a Art. 31.

Hence the focuses of incident and emergent rays at a prism, lye always very nearly at equal distances from it; provided the refractions and the refracting angle be but small. For then the perpendiculars TC, Tc are nearly equal; and in glass QC and qc are two thirds of them respectively.

Fig. 110.

Hence when the planes AB, ab are parallel, TC and Tc coincide; and Qq is one third of Cc, the thickness of the glass.

Images formed by a flat piece of glass. Fig. 111, 112.

53. An image pqr, formed by a flat piece of glass $AB\,ba$ is upright, parallel and equal to the object, PQR, and lyes on the same side of the glass as the object; but nearer to it by a third part of the thickness of the glass: because we have shewn that the focuses p, q, r, of the several pencils that flow from P, Q, R, lye so much nearer; in the lines PA, QC, RB, drawn from the several points of the object, perpendicular to the glass.

Images formed by a prism. Fig. 113.

54. An image formed by a prism is always upright, and equal to the object, and lyes on the same side of the prism, and at the same distance from it as the object it self: provided the refracting angle of the prism, and the refractions made by it, be but small. Take two rays, PE, QE, which coming from the extremities of the object, pass through a point E, so near to the angular point of the refracting angle, that the distances between their points of incidence and emergence need not be mentioned. And since the total bendings of the rays PEN, QEO are equal[b], they will cross each other, so as that the angle PEQ will be equal to the angle NEO or to pEq, made by the emergent rays produced backwards: and because the distance Ep of the focus p, of the pencil that flowed from P, is equal to EP[c], and in like manner the focal distance Eq equal to EQ; the image pq will be upright and equal to the object and at an equal distance on the same side of the prism.

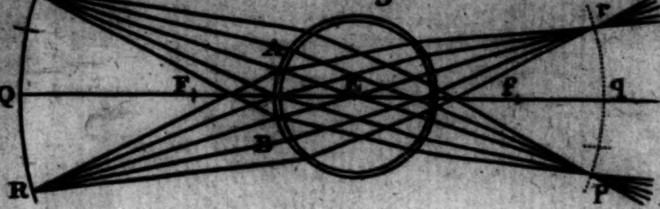
b Art. 40.

c Art. 25.

Fig. 114, 115, 116.

The same things might have been proved, by conceiving two rays PA, QB to go from the extremities of the object either parallel to one

123.



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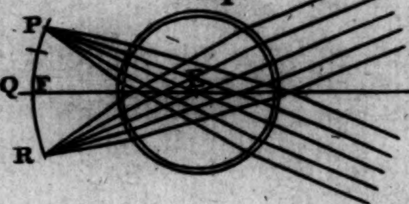

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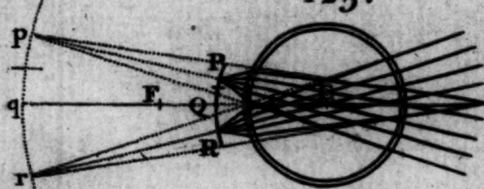

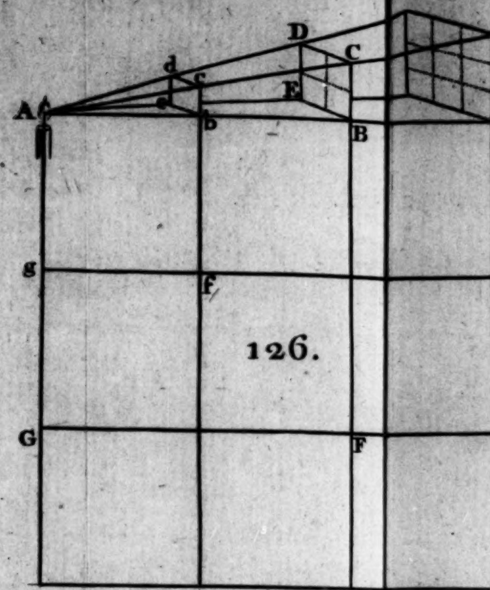

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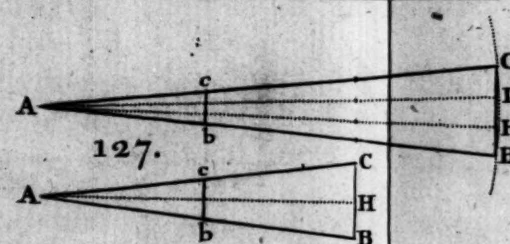

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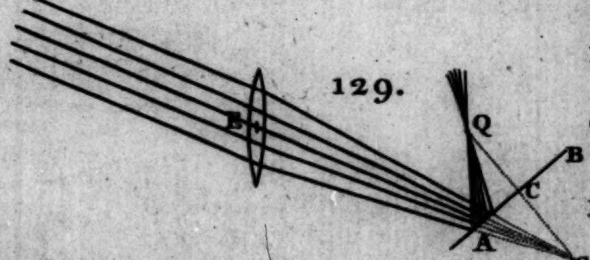

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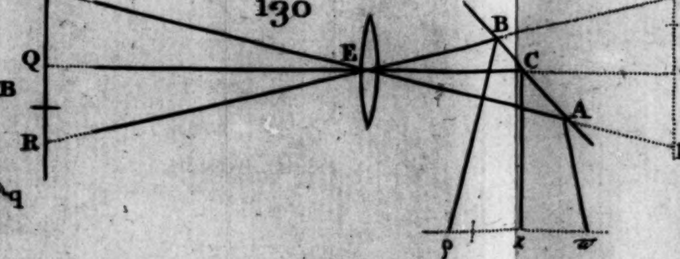

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one another or ſo as to croſs at any point E. Becauſe the emergent rays (produced) will accordingly be parallel [a], or will croſs at another point e on the ſame ſide of the priſm, and at the ſame diſtance from it as E [b]; and will alſo make equal angles at E and e, as they did when they croſſed at the angular point of the priſm [c].

a Art. 30.
b Art. 52.
c Art. 40.

Images formed by glaſſes of all ſorts. Fig. 117 to 125.

55. Theſe figures repreſent the manner in which an image of an object is formed by ſeveral pencils refracted through a glaſs of any ſort. And becauſe the axes PEp, QEq, REr of the ſeveral pencils go, in a manner, ſtraight through the center of the glaſs, the properties of theſe images, are the ſame as of thoſe which are made by refraction or by reflection at a ſingle ſurface, and deſcribed in the 29th article. Except that the image of an object which touches a globe does not coincide with the object it ſelf, but is ſituated at a diſtance from it, for the reaſon given at the end of the 48th article. According to theory the image of a circular arch will be nearly circular [d]; but when the object and image are ſmall and are placed at a conſiderable diſtance from the glaſs, the difference in their figures will be inſenſible in phyſical matters whether they be conſidered as circular arches or as ſtraight lines. Eſpecially conſidering that all the rays of a pencil do not croſs preciſely in a ſingle point of its axis, but in ſeveral points which make up a ſenſible part of it; as will appear by the following experiments.

d Art. 49.

Reflections and refractions at rough unpoliſhed ſurfaces.

56. When rays of light fall upon any rough unpoliſhed ſurface of an opake or tranſparent body, they are not reflected or refracted regularly, according to the laws and properties of poliſhed ſurfaces, but are ſcattered every way alike from the inequalities of the rough ſurface in the ſame manner as if they were emitted from a ſelf-ſhining ſubſtance.

A deſcription of ſome eaſy experiments, by which the truth of the foregoing properties of glaſſes may be readily examined, and ſome others be diſcovered.

I. Experiment. To ſhew that the breadths of a pencil are as their diſtances from the focus. Fig. 126.

57. Let the light which flows from a point A and paſſes through a ſquare hole $bcde$ be received upon a plane, $BCDE$, parallel to the plane of the hole; or if you pleaſe let the figure BD be the ſhadow of the plane bd; and when the diſtance AB is double of Ab, the length and breadth of the ſhadow BD will each be double the length and breadth of the plane bd; and treble, when AB is treble of Ab; and ſo on: which may be eaſily examined by the light of a candle placed at A.

Hence the denſity and quantity of light received upon a given plane are reciprocally as the ſquares of

58. Therefore the ſurface of the ſhadow BD, at the diſtance AB double of Ab, is diviſible into four ſquares, and at a treble diſtance, into nine ſquares, ſeverally equal to the ſquare bd, as repreſented in the figure. The light then which falls upon the plane bd, being ſuffered to paſs to a double diſtance, will be uniformly ſpread over four times the

its distances from the luminous body.

the space, and consequently will be four times thinner in every part of that space, and at a treble distance it will be nine times thinner, and at a quadruple distance sixteen times thinner, than it was at first; and so on according to the increase of the square surfaces $bcde$, $BCDE$ &c, or of the square surfaces $Abfg$, $ABFG$, &c, built upon the distances Ab, AB, &c. Consequently the quantities of this rarified light received upon a surface of any given size and shape whatever, removed successively to those several distances, will be but one quarter, one ninth, one sixteenth, of the whole quantity received by it at the first distance Ab. Or in general words the densities and quantities of light, received upon any given plane, are diminished in the same proportion as the squares of the distances of that plane, from the luminous body, are increased: and on the contrary, are increased in the same proportion as those squares are diminished. For the lights of the several points of the body, which severally follow this rule, will compose a light which will still follow the same rule.

Equal parts of a small object subtend equal angles at the eye. Fig. 127.

59. When the perpendicular subtense BC of a small angle BAC is divided into any number of equal parts BH, HI, IC, the lines, HA, IA, drawn from the points of division to A, will divide the angle BAC into the same number of parts, which will be nearly equal among themselves. For they would be so exactly if the line BC was an arch of a circle, drawn upon the center A; from which it differs so much the less as the angle at A is smaller; and so the proposition is exactest in the smallest angles.

Small angles subtended by the same object are reciprocally as its distances from the eye. a Art. 57. b Art. 59.

60. When the distance AB is double or treble of Ab, the subtense BC will be double or treble of the subtense bc of the same angle at A[a]. Divide BC into its parts BH, HI, IC, each equal to bc, and the rays HA, IA, will divide the angle BAC into as many equal parts[b]. Therefore when two angles bAc, BAH are subtended by the same or by equal lines bc, BH, the magnitude of the first angle bAc, will be to the magnitude of the second BAH, as the second distance BA to the first distance bA.

II. Experiment. To measure the focal distance of a globe of water, and of glass. Fig. 107.

61. Take an hollow globe of glass, or instead of it a thin round flask or decanter, and making a moderate round hole, about an inch broad, in a piece of brown paper, paste it on one side of the belly of the decanter; and having filled it with water, hold the side that is covered, to the sun, that the rays falling perpendicularly upon the hole, may pass through the middle of the water; and the emergent rays will be collected to a focus, whose nearest distance from the decanter will be equal to a semidiameter of the belly of it: as will appear by receiving the rays upon a paper held at that distance. That this effect is owing to the refraction of the water and not at all to that of the glass shell, appears to be reasonable by the 37th article; and will appear to be fact, by trying the experi-

experiment over again with the empty decanter. For the light paſſing through the hole, being received upon a paper, will be as broad as the hole it ſelf at all diſtances of the paper from the decanter. If the like experiment be tryed with a ſolid globe or ball of glaſs, the diſtance of its focus from the neareſt part of the ball will be one quarter of its diameter.

62. Things remaining as before, paſte a piece of thin white paper upon the ſide of the decanter oppoſite to the hole in the brown paper; and when the light of the ſun, which comes through the round hole, falls upon the white paper, meaſure its breadth, *GH*, with a pair of compaſſes; and it will be nearly equal to half the breadth, *AB*, of the hole in the brown paper. Which ſhews that if the converging rays *AG*, *BH*, were permitted to go ſtraight forward in a body of water continued far enough, they would convene at a focus, *T*, whoſe diſtance *DT*, from the neareſt point of the ball, would be about half[a] of, *CT*, its diſtance from the remoteſt point; and conſequently would be equal to the diameter *CD*; and therefore *CT* is to *TE* as 4 to 3, as was ſaid in the 33d article. If the white paper be paſted on the back ſide of a ſolid glaſs ball, the diameter *GH* of the circle of light will be found equal to one third part of *AB*; conſequently the rays *AG*, *BH*, converge to a focus *T* whoſe diſtance from *D* is one third[b] of its diſtance from *C*; that is, *CT* is to *TD* as 3 to 1 and conſequently *CT* is to *TE* as 3 to 2, as was ſaid in the 33d article. If the experiment be tryed with a lighted candle placed at a great diſtance; while the candle approaches to the ball, the breadth of the ſpot *GH* will increaſe continually; which ſhews that the focus *T* goes from the ball, and ſo it confirms the 34th article.

III. Experiment. To meaſure it after the firſt refraction only.

a Art. 57.

b Art. 57.

63. Having covered either ſide of a convex glaſs with paper, in which there are ſeveral ſmall holes made with a pin, and having expoſed the glaſs directly to the ſun, the rays which paſs through the holes will appear like ſo many white ſpots upon a paper held pretty cloſe behind the glaſs; and theſe ſpots will come cloſer together as the paper is gradually drawn back from the glaſs till at laſt they all unite in one ſpot or focus. The diſtance of this focus from the glaſs may therefore be meaſured, and will not be ſenſibly altered by turning the other ſide of the glaſs to the ſun[c], nor by inclining it a little to the incident rays[d]; and provided this ſmall inclination be ſo made as not to move the middle point of the glaſs, the focus or ſpot upon the paper will not be ſenſibly moved. Which ſhews that the axis of the oblique pencil continues ſtraight as before[e]. If the paper be drawn farther from the glaſs, the ſpots will recede from each other.

IV. Experiment. To meaſure the focal diſtance of a convex lens. Fig. 105.

c Art. 47.
d Art. 50.
e Art. 43.

64. If a concave lens covered in like manner be expoſed to the ſun, the ſpots of light which come through the holes and fall upon the paper behind the glaſs, will continually recede from each other as the

V. Experiment. To meaſure the focal diſtance of a concave lens.

 paper

paper is gradually removed from the glaſs. Which ſhews that the emergent rays continually diverge from a focus ſituated before the glaſs. Fig. 106. When the diſtance ab, of any two ſpots from each other, is double the diſtance AB, of the two correſponding holes in the cover, through which they came; the diſtance Ef, between the paper and the glaſs, is a Art. 57. then equal to its focal diſtance EF[a]; and by this means it may be meaſured.

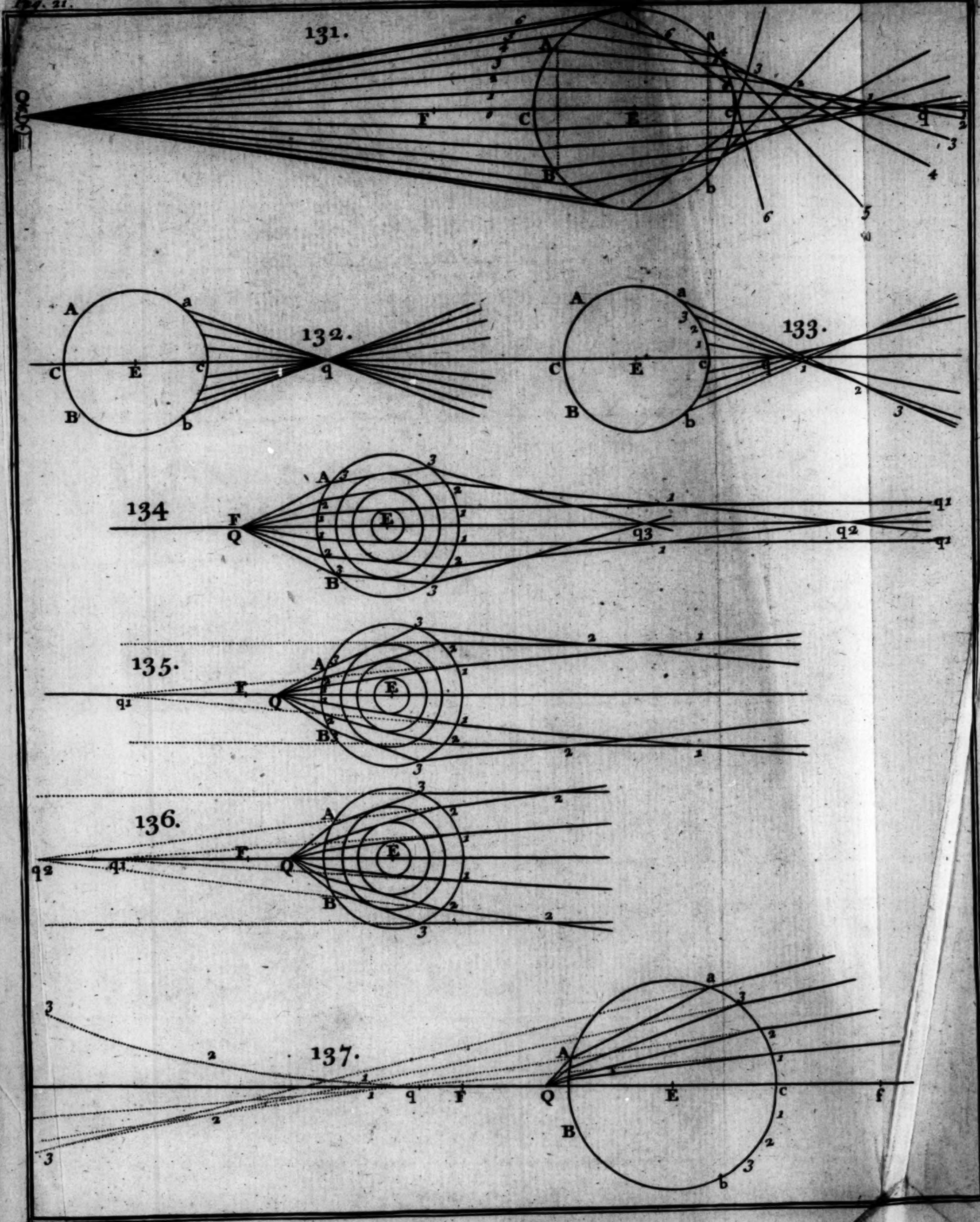
By theſe experiments it will be found, that the focal diſtance, EF, of a plano-convex or of a plano-concave glaſs is equal to a diameter of its convex or concave ſurface, that is, of the whole ſphere it belongs to: which proves the 33d article by holding the plane ſide of the glaſs perpendicular to the incident rays, that they may paſs through it unrefracted. Secondly, that the focal diſtance, EF, of a double-convex or double-concave glaſs, of equal convexities or concavities, is equal to a ſemidiameter of either of its ſurfaces: and conſequently that the focal diſtance of a glaſs of unequal convexities or unequal concavities will have an intermediate length between a diameter and a ſemidiameter of that ſurface which is moſt convex or moſt concave. For if a glaſs of equal convexities or concavities be conceived to grow gradually flatter on either ſide till it becomes quite flat, its focal diſtance will grow gradually b Art. 40, 41. longer[b], till at laſt it becomes a diameter of the remaining ſurface, as was ſaid above.

The like experiments may be tryed with a concave or a convex looking-glaſs covered with a paper ſtuck full of holes, to confirm the 26th article.

VI. Experiment. To ſhew the relation of conjugate focuſes of a lens to its principal focuſes. Fig. 128.

65. Having found the focal diſtance, EF, of a convex glaſs and fixed it flat againſt a moderate hole made in a thin board CE, placed upright upon a long table or floor; through the point C, directly under the middle of the glaſs, draw a long line AB perpendicular to the board; in which meaſure the focal diſtance of the glaſs from C to F and from F to I, I to II, II to III, &c; and alſo on the other ſide from C to f and from f to 1, 1 to 2, 2 to 3, &c; then taking $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the focal diſtance, ſet them off from F towards I and alſo from f towards 1, and put the figures $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, to the points of diviſion, as in the ſcheme; laſtly having darkened the room, if a candle be placed at Q over the mark I, the rays which paſs through the glaſs will be united at q upon a paper held over the oppoſite mark 1; and removing the candle to II and the paper to $\frac{1}{2}$, the rays will be united here alſo; and likewiſe when the candle and paper are removed to III and $\frac{1}{3}$, IV and $\frac{1}{4}$ &c: and the effect will be the ſame if the paper and candle be tranſpoſed into each others places: which confirms the 48th article. Beſides, it appears that fq varies reciprocally as FQ varies; that is, it decreaſes in the ſame proportion as FQ increaſes, and on the contrary.

66. Things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66. Things remaining as they were, when a ſecond candle is placed on either ſide of the firſt at the ſame diſtance from the glaſs, the union of its rays will make another image upon the paper q, on the contrary ſide of the axis QEq; and the diſtance between the two images will be found to bear the ſame proportion to the diſtance between the candles, as the diſtance of the images from the glaſs, bears to the diſtance of the candles from the glaſs. Theſe obſervations confirm the reaſon why the image of a ſingle candle is inverted upon the paper; and why its magnitude is altered when its place is altered. Becauſe what has been obſerved of two candles is applicable to any two points of the ſame candle; ſo that the deſcriptions of images in the 55th article is ſufficiently illuſtrated by this experiment. And what has been tryed with a convex lens may alſo be tryed with a concave looking-glaſs placed at the hole in the board.

VII. Experiment. To ſhew the inverſion and magnitude of an image.

67. If the rays of the ſun or moon or of a remote candle, which are made to converge to a focus q by a convex lens E, be intercepted by a looking-glaſs AB, they will be ſo reflected from it, as to converge to a focus Q at an equal diſtance before the looking-glaſs to that of q behind it. This may be examined by holding a piece of white paper at Q to receive the reflected rays. Therefore if the reflected rays be ſuppoſed to go directly backward, that is, from Q towards the looking-glaſs, AB, they will be reflected from it ſo as to diverge from q; which proves the 23d and 24th articles. If the convex-glaſs be placed in a hole of a window-ſhutter, and the room be darkened, the 130th figure ſhews how images of external objects, as PQR, which are ſeen inverted upon a paper held upright at pqr, may, by reflecting them downwards upon a paper held horizontally at $\varpi\kappa\varrho$, be viewed upright, when the ſpectator's back is turned towards the lens.

VIII. Experiment. To make images of objects appear upright in a dark room. Fig. 129.

68. Whatever be the ſhape and magnitude of the hole in the paper that covers part of a lens, the ſhape and magnitude of the picture of an object will be the ſame as when the lens is uncovered; becauſe any ſmall part of a pencil of rays has the ſame focus as the whole: but the brightneſs of the picture will be diminiſhed in proportion as the hole in the cover is diminiſhed; becauſe the quantity of light which illuminates every point of the picture is diminiſhed in that proportion. If the lens be very thick and broad, by this diminution of its aperture the diſtinctneſs of the picture may be ſenſibly improved. Becauſe the rays which fall upon the margin of ſuch a glaſs are not refracted exactly to the ſame points as thoſe which fall nearer the middle of it: which will be manifeſt by the next experiment.

IX. Experiment. To ſhew the degrees of brightneſs and diſtinctneſs of an image.

69. When the light of a candle or of the ſun is refracted through a globe, or through the belly of a round decanter filled with water, and falls upon a table-cloth, or upon a piece of white paper held parallel and very

X. Experiment. A cauſtick what and how formed by a globe or by a cylinder. Fig. 131.

very near to the axis of the light; the luminous figure there formed is bounded by two bright curves, called caufticks; which in going from the globe approach towards one another and to the axis of the pencil, till they touch it and there make a fharp angle, whofe point is the focus of the pencil.

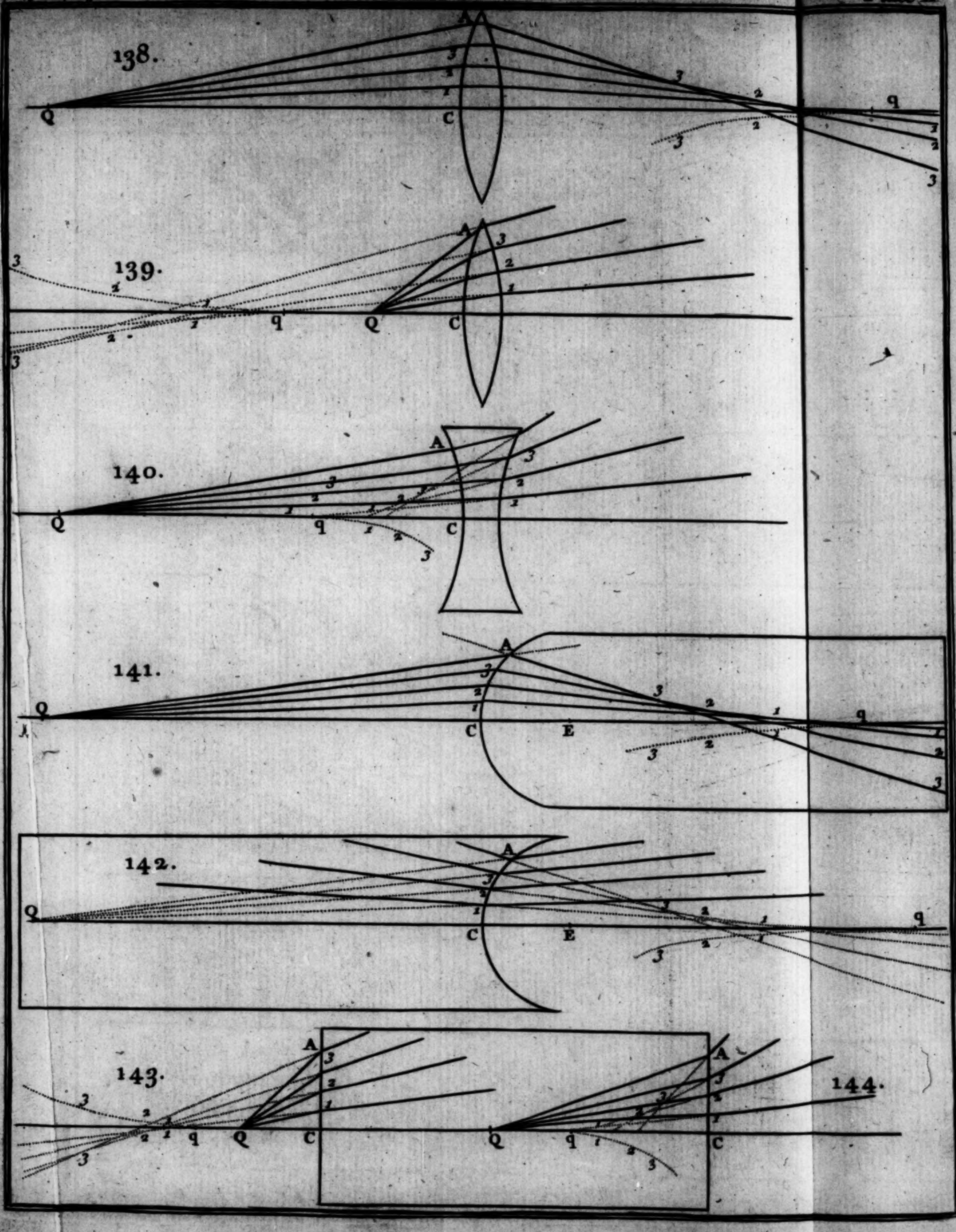
From the brightnefs of thefe curves it will appear, by infpection of the figure, that they are formed by the fucceffive interfections of every ray with the next to it, taken in fucceffive order from one fide of the globe to the other; and confequently that the brightnefs of the paper within the curves, and its darknefs without, is caufed by a multitude of interfections of rays within, and by none at all without.

It will alfo appear by the figure and pofition of the cauftick, that every ray croffes the next ray in a point of the cauftick before it cuts the axis. For if every ray croffed the next ray in a point of the axis, they muft all crofs it in one and the fame point; and fo the figure of the light upon the paper would confift of two bright angular fpaces, bounded not by curves but by ftraight lines that crofs in the focus; and confequently each angular fpace, at equal diftances on each fide of the focus, would be equally bright; which is contrary to experience. (Fig. 132.)

And if every ray croffed the next ray, after it had cut the axis, their fucceffive interfections would form a bright curve, which would have a fharp angle at the focus as before; but would diverge farther and farther from the axis in going from the globe, as reprefented in the figure; which is alfo contrary to experience. It is manifeft then from the fhape and pofition of the cauftick, that every ray croffes the next ray before it cuts the axis; and alfo that the focus of the pencil is that point of the axis where the neareft rays cut it; and that the rays in the incident pencil, which lye farther and farther from the axis, cut it in feveral points, that lye farther and farther from the focus. (Fig. 133. Fig 131.)

An imaginary cauftick formed by a globe or by a cylinder. Fig. 134. Fig. 135. Fig. 136. Fig. 137.

70. Therefore fince the total bending of a ray is not altered, while it paffes at equal diftances from the center of the globe, and confequently touches a circle concentrick to it, it follows, that while the candle is moved gradually towards the globe, the neareft rays to its center, on oppofite fides of it, will firft become parallel to the axis, and prefently after will diverge from a point behind the candle; then the next rays to thefe, on oppofite fides of the center, will alfo become parallel to the axis, and after that will diverge from another point of it, fomewhat farther behind the candle than the former point, from which the neareft rays diverged; and fo on. Confequently when the emergent rays diverge, each contiguous couple being produced backwards, will cut the axis before they crofs one another; and thefe fucceffive interfections, from whence each couple diverge, will form an imaginary cauftick, beginning with a fharp angle at the focus, and receding from the axis, in going backwards from the globe.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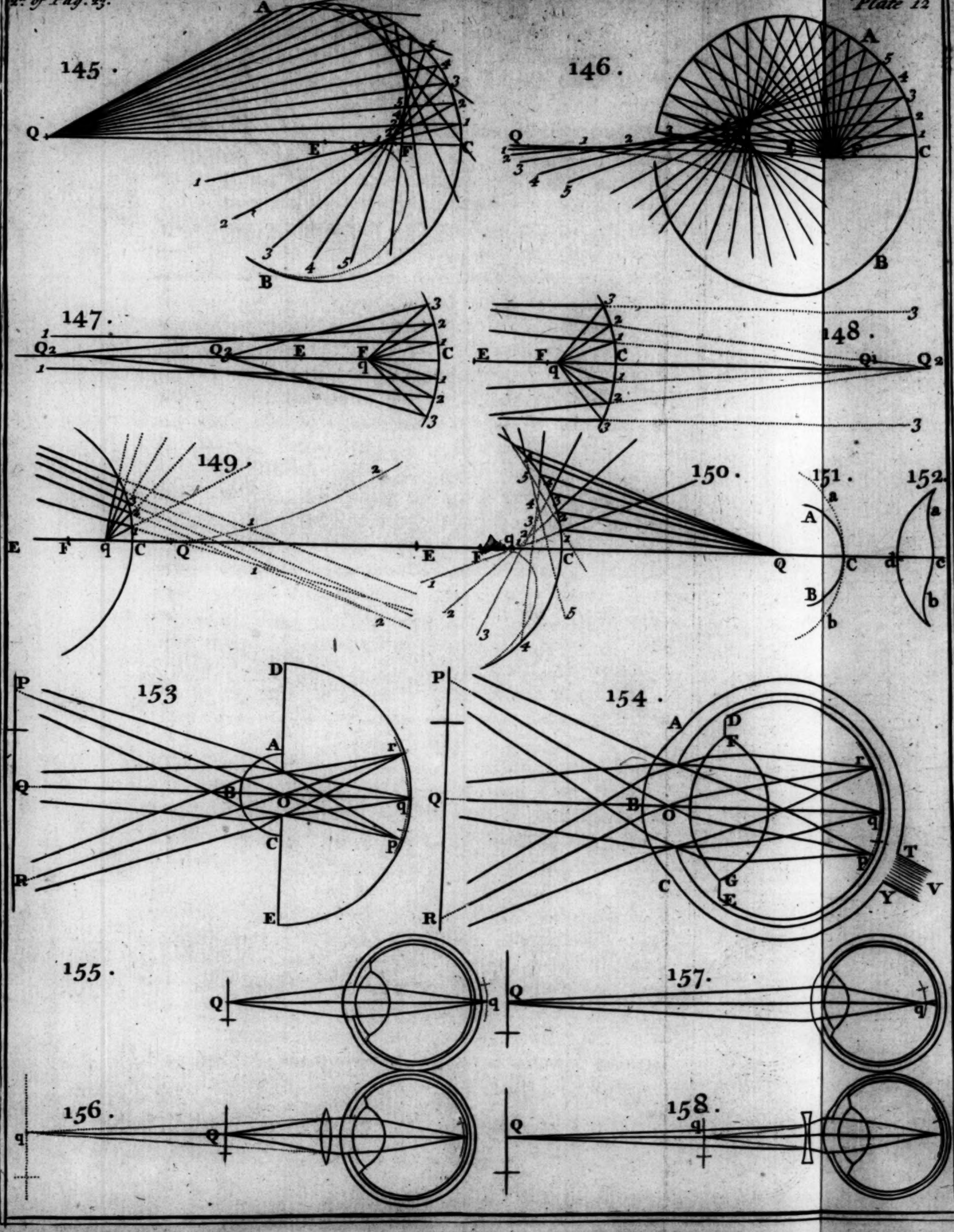
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71. A large pencil of rays refracted through a convex lens is also formed into a caustick, or some part of a caustick adjoining to the focus of the lens; and extending from it more or less according as the lens is composed of larger or lesser segments of the spheres whose convexities it has. For conceiving two planes *AB, ab* to cut off two opposite segments *ACB, acb* from the globe, where the rays passed through it, their refractions through the segments, when joined together, will be much the same as when they were separated by the middle part of the globe that lay between them; and consequently the causticks formed by the lens and by the globe will have like properties.

Causticks formed by a convex lens. Fig. 138, 139.

Fig. 131.

The truth of this description of causticks, may be farther confirmed by covering one side of the globe or of a broad convex lens, with a large circle of brown paper, whose diameter has a row of pin-holes made in it at any equal distances from one another. For the spots of light which came through these holes, will appear upon a white paper, at equal distances from one another, when the paper is held perpendicular to the rays and near to the glass. But while it is withdrawn from the glass, the intervals between the exteriour spots will grow less than the intervals between the interiour, and will sooner unite.

72. On the contrary if the same cover be put upon a concave lens, while the paper is drawn from the lens, the intervals between the exteriour spots will grow larger than the intervals between the interiour: which shews that the exteriour rays diverge from points that are nearer the concave, than those from whence the interiour rays diverge. But this experiment will not succeed with common concave glasses made for short-sighted persons: they are not concave enough, nor broad enough, nor thick enough to make this effect sensible.

And by a concave lens. Fig. 140.

73. It appears by these real and imaginary causticks, that the exteriour rays of a pencil are gradually too much bent; or, which is the same thing, the interiour rays are gradually too little bent, to belong all together to a single point after refraction; and consequently the angles of incidence of the exteriour rays are too large for that purpose, both at the first and second surface of the globe or lens.

The bendings of the exteriour rays too great.

74. Consequently the like causticks must be formed by the refractions of a pencil of rays at a single surface; only these causticks will approach slower towards the axis than the former, or recede from it slower; because every couple of contiguous rays has now but a single refraction to make them converge, or but a single refraction to make them diverge.

Causticks formed by refraction at a single spherical surface. Fig. 141, 142.

75. And it is demonstrated in the next book, that the rays of a large pencil, refracted by a single plane surface, will also diverge from the points of an imaginary caustick; which begins at their focus and recedes from the surface, when the refractions are made out of a rarer into a denser medium; and accedes towards it, when they are made out of a denser into a rarer.

And by a plane surface. Fig. 143, 144.

76. The

A caustick formed by reflection at a spherical or cylindrical concave surface.

76. The 145th figure represents the shape of a caustick formed by the successive intersections of the contiguous rays of a large pencil, reflected from a concave surface, either spherical or cylindrical. Such causticks may be seen upon the surface of milk, or upon any opake whitish mixture of liquors contained in a white china-cup, or upon the bottom of a snuff-box, whose rim is well polished, when the light of a candle or of the sun or of a remote window shines upon it.

Another caustick formed beyond the center of the concave. Fig. 145, 146. a Art. 27.

77. While the points of incidence keep fixt, imagine all the lines described by the reflected rays to accede towards the center, till they unite at the focus of the pencil; and supposing the rays to go backwards in the same lines, after this second reflection they will all recede from their former focus and accede towards the contrary side of the center [a]: and the exteriour rays, whose first intersections with the axis were farthest from the center, will now come nearest to the opposite side of it [a]. So that when the luminous point is placed between the principal focus and the center, another caustick will be formed beyond the center.

An imaginary caustick by reflection from the concave. Fig. 146, 147, 148, 149.

78. Therefore while this luminous point is moved gradually towards the surface, when it comes to the principal focus, the rays that are nearest to the axis will first become parallel to it, and presently after will diverge from a point behind the concave; then the next rays to these will also become parallel to the axis, and after that will diverge from another point of it, somewhat farther behind the surface than the former, from which the nearest rays diverged. Consequently each couple of reflected rays, which lye contiguous, being produced backwards, will cut the axis before they meet one another. And these successive intersections, from whence each couple diverge, will form an imaginary caustick behind the concave, beginning with a sharp angle at the focus and receding from the axis in going from the surface.

An imaginary caustick by reflection from the convex side of the surface.

79. While the points of incidence keep fixt imagine all the intersections at the axis in the 149th figure to be shoved to the focus Q, as in the 150th figure, and let rays diverge from it upon the convex side of the surface; and the reflected rays produced will all separate from the former focus q, so as to form an imaginary caustick, and the remotest rays from the axis, whose intersections were shoved the farthest towards the surface, will accede the nearest to it after reflection [b].

b Art. 27.

A general observation upon causticks.

80. In all these causticks by refraction and reflection at plane and spherical surfaces, the concourse of two contiguous rays (produced) lyes farther from the focus, and from the axis, according as their points of incidence are farther from the axis. It is observable that a pencil of rays reflected from a plane surface will form no caustick at all, because they diverge accurately from a single point [c].

c Art. 23.

No spherical figure can refract or reflect

81. From what has been said it appears that a spherical surface, having every where the same degree of curvity, can neither reflect nor refract

fract all the rays of a large pencil to a single point; and that a single surface fit for this purpose must grow straighter or flatter gradually in going from its axis[a], as represented in the 151st figure; and that if one side of a lens be spherical, it is not sufficient that the other be plane, but it must be convex in the middle, to shorten the focal distance of the middlemost rays, and must become concave towards its circumference, to prolong the concurrence of the outermost rays; as represented in the 152d figure. Nevertheless the middlemost rays of a pencil are crowded so close together by reflection and refraction at spherical surfaces and lenses, and the outermost rays are scattered so thin upon a plane passing through the focus perpendicular to the axis, that the confusion they make in a picture, by mixing with rays of other pencils, is seldom sensible when the glass has a moderate aperture. And since the unequal degrees of refrangibility of rays of different colours (to be explained in the 6th chapter) makes far greater aberrations from the focus, than those that are made by the sphericalness of the figure, it would be to little purpose to give glasses any other figure than spherical; especially considering the great difficulty there would be in the mechanical operation.

rays to a single point.

a Art. 73, 76.

CHAP. III.

CONCERNING THE EYE AND MANNER OF VISION.

82. CONSIDERING what has been said in the 33d and 35th articles, one might contrive a tolerable eye in this manner, by placing a pellucid hemisphere ABC to serve for the fore part, and another concentrick one DqE, opposite to the former, to serve for its bottom or back part; making the semidiameter, Oq, of the latter triple the semidiameter, OB, of the former; and then by filling the whole cavity of both with water. By this means rays of light flowing from the points P, Q, R, &c, of remote objects, after refraction at the surface ABC, will be collected to as many other points p, q, r, of the cavity DqE, and paint an image upon it. And because a spherical surface does not accurately refract all the rays of a large pencil to a single point[b], but only those that go pretty near its axis; this imperfection might be remedied by covering the base AC, of the lesser hemisphere, all but a moderate hole about the center O; which would answer the purpose much better than if the surface it self was covered, all but a hole in the middle about B. For in this latter case the surface ABC would not receive rays from the lateral points P, R, so directly as those from the middle of the object, to all which it is exposed alike when the hole is left open at the center O.

A fictitious eye described by *Hugens**.

Fig. 153.

b Art. 81.

* Opuscula Postuma. p. 112.

And compared to the natural eye.

83. Though this construction of the eye appears not amiss at first sight, yet we shall see presently that the author of nature has wisely varied some things for the better, and added others absolutely necessary; though in every thing we cannot perceive his designs. In the first place he would not make use of an intire hemisphere *ABC*, but retaining the middle part, has taken off pretty much from the sides, and yet without contracting the compass of objects taken in at one view. The reason of this was to bend inwards the edges of the larger hemisphere about *D* and *E*, thereby reducing the shape of the eye to a rounder figure, for the convenience of its motion every way in the cavity that contains it. He has therefore given it such a shape, as is expressed in this other figure, representing an human eye dissected through its axis, all the parts being twice as big as in the life to render them more conspicuous.

Fig. 154.

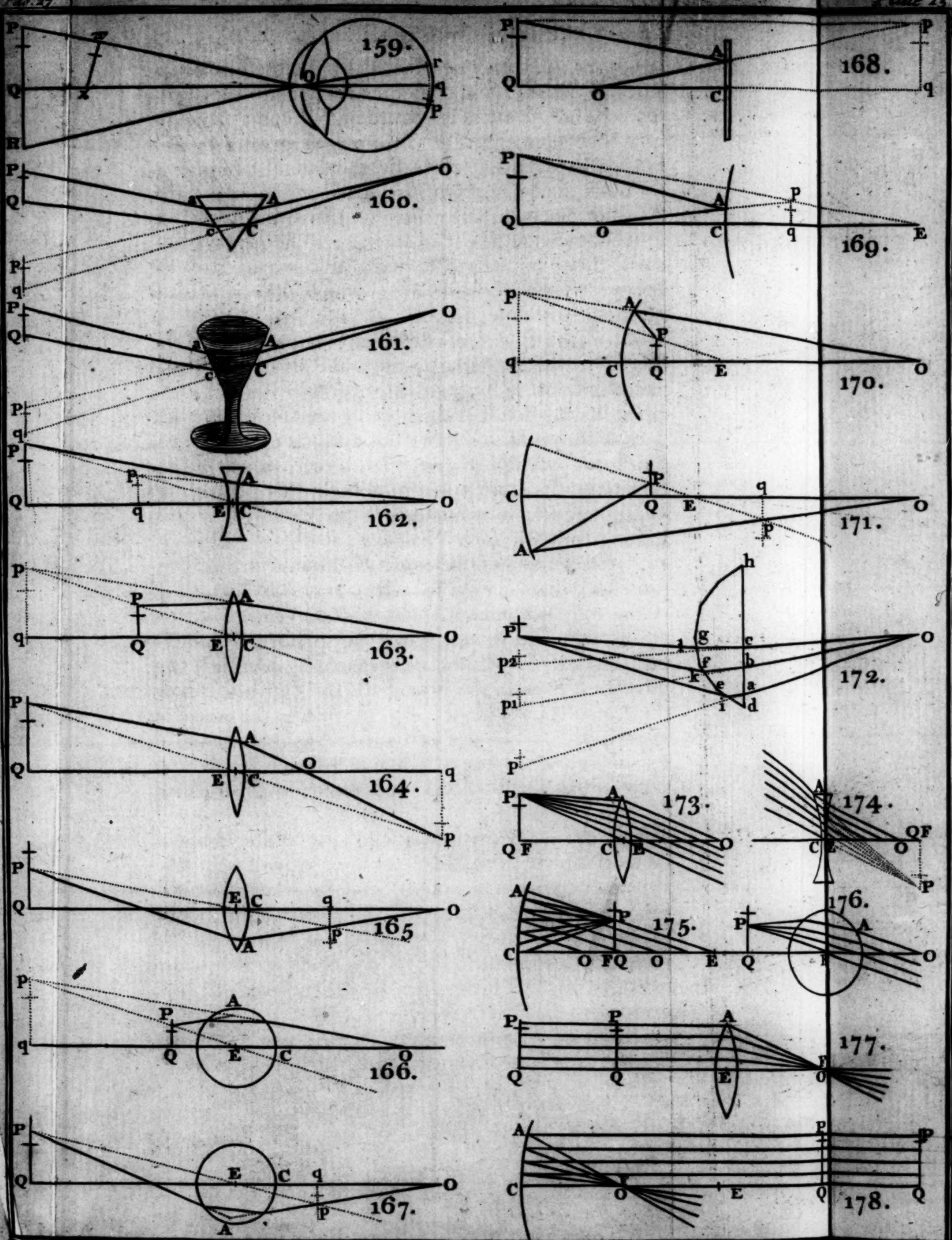
An human eye described.

84. Here the transparent parts of the coat called the cornea is *ABC*; the remainder *ATYC* being opake, and a portion of a larger sphere. Within this outward coat anatomists distinguish two others; the innermost of which is called the retina, being like a fine net composed of the fibres of the optick nerve *YVT* woven together, and is white about the parts *p*, *q*, *r*, at the bottom of the eye. The cavity of the eye is not filled with one liquor, but with three of different sorts. That contained in the outward space *ABCOEGFDO* is called the aqueous humor, being perfectly fluid like water; the other contained in the inward space *EpqrDFG* is a little thicker like the white of an egg, and is called the vitreous humor; the third humour *FG* is shaped like a lens of unequal convexities, lying between the two former, and fixed to the side coats by filaments or threads extended all round it, and is called the crystalline humor, being hard like the white of an egg boiled, but as clear as the other two, and differs from them in a greater degree of refractive power; whereby the rays that came from the points *P*, *Q*, *R*, having received a degree of convergence by the refraction of the cornea *ABC*, are made to converge a little more by other refractions at the surfaces of the crystalline *FG*; so that uniting in as many other points *p*, *q*, *r*, upon the retina, they represent the points of the object *P*, *Q*, *R* from whence they came. And perhaps the rays are so directed by these secondary refractions at the crystalline, as to fit the cavity *pqr* intended to receive them; which otherwise must have been a portion of a larger sphere[a], according to the fictitious design in the former figure.

a Art. 73, 80.

The crystalline makes all pictures distinct.

85. Besides this there was greater need of the lens *FG* upon another account; namely to help the eye to conform it self for seeing objects distinctly at all distances, which was wanting in the fictitious eye. There are two ways of doing it by the help of this lens *FG*, in order to see things near at hand; either by moving it nearer to the outward cornea, or by increasing its convexity, or perhaps by doing both at once. If it is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is moved towards the cornea, this may be effected by the pressure of the muscles against the sides of the eye, and consequently against the vitreous humor; but if the crystalline alters its figure and becomes rounder for seeing near objects, the filaments *DF*, *EG*, whose greater tension helps to flatten it, may perhaps be slackened by the lateral pressure aforesaid; and possibly both these alterations are made at the same time. The hole or pupil *O* is not placed in the center of the cornea *ABC*, as in the fictitious eye, but somewhat nearer to its front. The reason is uncertain, unless this also may contribute to make the images coincide with the cavity of the retina, (in all their parts,) which otherwise must have been shaped according to a larger sphere [a]. a Art. 73, 80.

Some dimensions of an human eye.

86. The diameter *AY* of the sphere of the eye is about an inch of the Rheinland foot, which is much the same as the old Roman foot: and the diameter of the outward cornea is about three fifths of an inch; the breadth of the pupil, *O*, has no fixt measure, being greater or smaller, as any one may try, according as less or more light shines upon the eye; it also contracts at the near approach of any small object, when we endeavour to view it distinctly. Its fabrick is admirable in this respect that while it alters its size it preserves its roundness [1]. So far Mr. *Hugens* [2], to which I add something from Sir *Isaac Newton* [3].

Pictures on the retina the cause of vision.

87. This account of the eye and of the cause of vision is farther confirmed by these arguments; that anatomists when they have taken off from the bottom of the eye that outward and thickest coat called the dura mater, can see through the thinner coats the pictures of objects lively painted thereon. And these pictures propagated by motion along the fibres of the optick nerves into the brain, are the cause of vision. For according as these pictures are perfect or imperfect, the object is seen perfectly or imperfectly. If the eye be tinged with any colour (as in the disease of the jaundice) so as to tinge the pictures in the bottom of the eye with that colour, then all objects appear tinged with the same colour.

Confused pictures in old mens eyes how caused, and mended by convex glasses. Fig. 155.

88. If the humours of the eye decay by old age so as by shrinking to make the cornea and coat of the crystalline humour grow flatter than before, the light will not be refracted enough, and for want of a sufficient refraction will not converge to the bottom of the eye, but to some place beyond it; and by consequence will paint in the bottom of the eye a confused picture; and according to the indistinctness of the picture the object will appear confused. This is the reason of the decay of sight in old men, and shews why their sight is mended by spectacles. For the convex glasses supply the defect of plumpness in the eye, and by increasing the refractions make the rays converge sooner, so as to convene di- Fig. 156.

1 See this accounted for in Mr. *Chesselden*'s anatomy p. 319. 3d. Ed.
2 Dioptrica prop. 31.
3 Opticks pag. 12.

ſtinctly at the bottom of the eye, if the glaſs has a due degree of convexity.

Confuſed pictures in ſhort-ſighted eyes, how cauſed and mended by concave glaſſes. Fig. 157. a Art. 34, 48. Fig. 158.

89. And the contrary happens in ſhort-ſighted men, whoſe eyes are too plump. For the refraction being now too great, the rays converge and convene in theſe eyes before they come at the bottom; and therefore the picture made in the bottom and the viſion cauſed thereby will not be diſtinct, unleſs the object be brought ſo near the eye as that the place where the converging rays convene may be removed to the bottom [a]; or that the plumpneſs of the eye be taken off and the refraction diminiſhed by a concave glaſs, of a due degree of concavity; or laſtly that by age the eye grows flatter till it comes to a due figure. For ſhort-ſighted men ſee remote objects beſt in old age, and therefore they are accounted to have the moſt laſting eyes. So far Sir *Iſaac Newton*.

The pupil of the eye may be conſidered as a point. Fig. 153. b Art. 91. c Art. 97.

90. In determining the magnitude of pictures upon the retina, only one ray in each pencil need be conſidered; becauſe when the picture is diſtinct, all the rays in any one pencil are collected to one and the ſame point of the retina. Or, which is much the ſame, we may ſuppoſe the pupil of the eye contracted to a point: and, for greater ſimplicity and eaſe of the imagination that this point O is a little hole at the center of a dark, hollow hemiſphere DqE, admitting only ſingle rays ſtraight through it without any refraction at all. For then the lengths of theſe pictures pqr will increaſe and decreaſe as the angle pOr does, or as the angle POR does; which I am going to ſhew to be the property of the natural eye [b]: and if the ſemidiameter Oq, of this hollow hemiſphere, be about $\frac{5}{8}$ of an inch, or of the axis of an human eye, the pictures of the ſame objects will always have the ſame bigneſs in both ſorts of eyes [c], very nearly.

Diameters of pictures on the retina are as the angles ſubtended by the object at the eye. Fig. 159.

91. The diameters or lengths of the pictures of objects upon the retina are meaſured by, or proportionable to, the angles which the rays that come from the extremities of the object do make in falling on the eye; provided thoſe angles be but ſmall. For let two or more objects PQ and $\varpi\varkappa$, either parallel or oblique to each other, ſubtend the ſame angle POQ or $\varpi O\varkappa$ at O; and becauſe the particles of light flowing from P and ϖ deſcribe the ſame line $P\varpi O$, they will be refracted to the ſame point p upon the retina; and in like manner thoſe that flow from Q and $\varkappa$ will be refracted to the ſame point q; and ſo the pictures pq of the objects PQ, $\varpi\varkappa$, which ſubtend the ſame angle at O, are the ſame in magnitude; which was the firſt thing to be proved.

d Art. 87.

Now the pictures of objects painted upon the retina of a dead eye are found by experience to be perfectly well ſhaped and proportioned in their parts [d]; that is, the proportion of the parts pq, qr, of the whole picture pqr, is the ſame as that of the parts PQ, QR, of the whole object PQR; and this latter proportion is very nearly the ſame as that of the

the angles POQ, QOR subtended by the parts PQ, QR [a]; and so the proposition is proved when the objects PQ, QR are both at the same distance from the eye. And since it was shewn just before, that the objects PQ and $\varpi\kappa$ have the same picture pq, it follows that the proportion of the pictures of the objects $\varpi\kappa$ and QR is the same as that of the angles $\varpi O\kappa$, QOR, subtended by them at the eye.

[a] Art. 59.

92. When an object approaches towards the eye, the diameter of its picture upon the retina increases in the same proportion as the distance between the eye and the object decreases; and on the contrary, it decreases in the same proportion as that distance increases. For the diameter of its picture increases in the same proportion as the angle increases, which the object subtends at the eye [b]; and this angle, when small, increases in the same proportion as the distance between the eye and object decreases [c].

Are reciprocally as the distances of the object from the eye.

[b] Art. 91.

[c] Art. 60.

93. The degree of brightness of the picture of an object painted upon the retina continues the same, at all distances between the eye and the object; provided none of the rays be stopt by the way, and that the pupil does not alter its aperture. For instance, when the eye approaches as near again to the object, the picture upon the retina becomes double in length and double in breadth, and consequently quadruple in surface; for the surface would be double, if its length alone or breadth alone was double. The quantity of rays received through the same aperture of the pupil, at half the distance from the object, is also quadruple [d]; and being equally spread over four times the quantity of surface of the retina, they are just as dense as before when the object was at twice the distance.

Brightness of pictures not altered by the distance of the object from the eye.

[d] Art. 58.

94. It follows then that the faint appearance of remote objects is owing to the opacity of the atmosphere, which hinders part of their light from coming to the eye. Accordingly we find that the sun, moon and stars appear very faint when near the horizon, and brighter continually as they rise higher; because the tract of vapours, which lies in the way of the rays, is longest and thickest near the horizon; and becomes thinner and shorter as the objects rise higher, and consequently does less obstruct the passage of the rays.

Faintness of pictures of remote objects how caused.

95. The sensibility of the eye, or its power to discern objects, without inconvenience, by different quantities of light, is vastly extensive. For instance, the disproportion in the quantities of light, cast upon the horizon by the sun and moon, at any equal altitudes, I find is no less than 90 thousand to 1, when the moon is full; or no less than 180 thousand to 1, when the moon is in the quarters. And the proportion between those parts of the lights of the sun and moon, whatever they be, which are reflected to our eyes from the same object by day and by night, can hardly be different from the proportion of the whole lights. Allowing then

Their degrees of brightness by day-light and moon-light compared.

then that the aperture of the pupil may poſſibly be 8 or 9 times leſs by day than by night, (that is about 3 times leſs in diameter,) yet the proportion in the quantities of day-light and moon-light, received by the eye from the ſame object, to illuminate a picture of the ſame bigneſs, will be no leſs than 20 thouſand to 1, when the nights have a middle degree of moon-light; I ſay no leſs, becauſe the numbers here given are deduced from a rule, which is built upon this principle; that the moon reflects all the light received from the ſun; which cannot be true, by reaſon of the appearance of very large obſcure places in her body; and in all probability a great part of the incident light is buried and loſt even in the brighteſt places.

The rule I mentioned is this, day-light is to moon-light as the ſurface of an hemiſphere, whoſe center is at the eye, to the part of that ſurface which appears to be poſſeſſed by the enlightened part of the moon: ſo that the whole heavens covered with moons would only make day-light. This will be evident enough from the following conſiderations, though I invented it another way. Day-light is made by innumerable reflections of the ſun's rays from all ſorts of bodies till at laſt they come to our eyes: for if this were not ſo, we could ſee nothing in the world, even in the day time, but the ſun and ſtars and ſelf-ſhining ſubſtances[a]. Accordingly we find that day-light is much the ſame, whether the ſun ſhines out or not, in the place we are in; becauſe his light is reflected to us from a vaſt quantity of earth, air and clouds extended all round us, perhaps to a hundred miles or more. So that the abſence of the ſun's rays from a particular place ſcarce alters day-light. Another thing is that the moon by day appears like a cloud in the air of a middle degree of brightneſs; ſome appearing duller and ſome brighter than the moon it ſelf. The rays of the ſun being therefore intercepted in the night from all the viſible clouds, and being reflected to us by the moon only, it follows that day-light is to moon-light, as the apparent ſurfaces of all the viſible clouds, to the apparent ſurface of the viſible part of the moon, conſidered as the only cloud which remains enlightened. And theſe two lights, whatever be the diſtances of the moon and clouds, are juſt the ſame as if thoſe bodies were all placed at any equal diſtances from us, and compoſed the ſurface of an hemiſphere[b]; whoſe parts are the true meaſures of the parts of the light which comes to us.

a Art. 2.

b Art. 93.

And confirmed by experiments with burning-glaſſes.

96. A vaſt diſproportion between the lights of the ſun and moon appears alſo by experiments made with burning-glaſſes; either by refraction of the rays through very broad lenſes, or by reflection from very broad concave-glaſſes or metals: which by collecting the rays of the ſun into a ſmall round image at the focus, do excite a more violent heat and burn quicker than the hotteſt wind-furnaces: as appears by their melting and calcining the hardeſt metals, and by vitrifying bricks and ſtones, in

in much leſs portions of time than a minute [1]. Yet the rays of the moon being collected by the ſame glaſſes, do not excite the leaſt ſenſible heat; nor do they ſenſibly affect the niceſt thermometer, when caſt upon the ball of it [2], though the brightneſs of the light be very ſenſibly increaſed. By meaſuring the breadth of the round image at the focus, and by comparing it with the breadth of the glaſs it ſelf, it appears that ſome of theſe burning-glaſſes collect the incident rays into a ſpace about 2 thouſand times leſs than they poſſeſſed at their incidence. But by the preceding calculation, the light of the full moon muſt be condenſed about 90 thouſand times[a], to make it as denſe and as warm as the direct rays of the ſun. It is no wonder then that the heat of the moon's rays is not ſenſible in the focus of the glaſs, being then even 40 or 50 times thinner than the direct rays of the ſun. For it is found by experiments made with theſe glaſſes that the degrees of heat are proportionable to the denſities of the rays: which being compared with a ſcale of the degrees of heat and warmth of ſeveral natural bodies, determined by Sir *Iſaac Newton*, in the philoſophical tranſactions [3], it appears there is a vaſt diſproportion between the degrees of light which the eye can bear and be ſenſible of, and the degrees of its heat which the touch can bear and be ſenſible of.

a Art. 95.

Viſion limited both by magnitude and diſtance.

97. Dr. *Hook* aſſures us that the ſharpeſt eye cannot well diſtinguiſh any diſtance in the heavens, ſuppoſe a ſpot of the moon's body, or the diſtance of two ſtars, which ſubtends a leſs angle at the eye than half a minute; and that hardly one of a hundred can diſtinguiſh it when it ſubtends a minute [4]. If the angle be not bigger than this, the two ſtars appear to the naked eye, as if they were but one. I have been preſent at making the experiment, when a friend of mine, who had the beſt eyes of all the company, could ſcarce perceive a white circle upon a black ground, or a black circle upon a white ground, or againſt the ſky-light, when it ſubtended a leſs angle at the eye than two thirds of a minute; or which is the ſame thing, when its diſtance from the eye exceeded 5156 times its own diameter: which agrees well enough with Dr. *Hook*'s obſervation. Hence I find, by a rule in the next book, that the diameter of the picture of that circle upon the retina was but the 8000th part of an inch at moſt; and this may be called a ſenſible point of the retina. That this point is very ſmall any one may perceive from hence that the breadth of the fineſt hair is viſible at the length of ones arm.

Apparent magnitude to the naked eye defined.

98. The apparent magnitude of an object is a quantity of viſible extenſion, meaſured by, or proportionable to, the angle which the two [b] rays

b Art. 90.

1 Phil. Tranſ. abr. by *Lowth*. Vol. 1. p. 211. and by *Jones* Vol 4. p. 190.
2 Ibid. *Lowth*. p. 213. and Mem. de l'Acad. Roy. des Scien. ann. 1705. p. 455. 8°.
3 N°. 270. or *Jones*'s abr. Vol. 4. part. 2. p. 1. It. Mem. de l'Acad. an. 1703. p. 233.
4 Animadverſions on Hevelii machina cœleſtis p. 8.

that

that come from the extremities of the object do make in falling on the eye. For the extremities of the object are seen in the directions of these rays; and in proportion as they make a greater or smaller angle at the eye, the magnitude of the picture upon the retina is longer or shorter [a]; and consequently causes a sensation of a greater or smaller visible extension; consisting of a greater or smaller number of visible points, answering to the number of sensible points of the retina [b], of what magnitude soever these points are supposed to be.

a Art. 91.

b Art. 97.

How it varies.

99. The apparent magnitude of any given object is reciprocally as its distance from the eye: that is to say when the object approaches to the eye, its apparent magnitude increases (in proportion) as its real distance decreases; and on the contrary, it decreases (in proportion) as that distance increases. For the apparent magnitude of an object was defined to be a quantity of visible extension proportionable to the angle which the object subtends at the eye [c], and this angle increases very nearly in proportion as the real distance between the eye and object decreases [d].

c Art. 98.

d Art. 60.

When called true magnitude.

100. The apparent magnitude of an object seen by the naked eye, in opposition to its apparent magnitude seen through glasses, and for shortness of expression, is often called its true magnitude. And in speaking of the apparent magnitude of an object I always mean the apparent magnitude of its diameter, or of its length or breadth, or of any principal line of it, and not of its surface or solidity, unless it be particularly specified.

CHAP. IV.

CONCERNING VISION WITH GLASSES.

Apparent directions of visible points defined.

101. ANY small object or point of an object, seen by refracted or reflected rays, appears somewhere in the direction of that line, which the visual ray describes after its last refraction or reflection in falling upon the eye.

e Art. 18. Fig. 4.

In the experiments to prove the laws of reflection and refraction [e], the pin at *B*, seen by a ray reflected from the water, appeared somewhere in the line *AC* produced, which the visual ray *BCA* described after reflection at *C*, when it advanced to the eye. And as the whole line *CE* appeared lifted up by the refraction at the water, as if it had been a continuation of the line *AC* straight on, so if a straight oar be in part immersed obliquely in water, it appears crooked, as if the part immersed was broken at the surface, and lifted higher. For this part of the oar is seen in the direction of rays which are bent downwards by refraction at their emergence from the water, and consequently advance to the eye as if they came from a place in the water which is higher than the real place of the oar. In like manner any point *P* of an object seen by the

the ray PAO twice refracted, either by passing through the edge of a prism, or of a concave or convex lens, or through the sides of a globe or decanter, or of a drinking glass filled with any transparent liquor; or seen by a ray PAO reflected from a plane or spherical looking-glass, appears to the eye at O, somewhere in the direction of the last refracted or reflected ray AO. Lastly an object P viewed by the eye at O, through a multiplying glass, appears at one view in as many different places, p, $p1$, $p2$, situated in as many different directions OA, OB, OC of the last refracted rays produced, as the glass has different surfaces DE, EF, FG differently inclined to the opposite surface DH. For these surfaces, like so many different prisms, give the visual rays $PIAO$, $PKBO$, $PLCO$ so many different bendings at I and A, K and B, L and C, and make them fall upon the eye at O in as many different directions AO, BO, CO [a]. And in all these instances when the reflecting or refracting surfaces of the water or glasses are shaken by the wind, or otherwise, the objects seen by reflection or refraction appear to shake and tremble; because the last directions of the visual rays are shaken and varied by those motions.

Fig. 160 to 171.

Fig. 172.

a Art. 40.

Now the reason why an object or point of an object appears always in the direction of the last refracted or reflected ray, is, because the place of its picture upon the retina is the same as it would be if the object was really removed from its proper place into the direction of that ray, and was seen by direct rays. And having no sensation of the previous reflections or refractions of the rays at the glasses, but only of their action upon a certain place of the retina, we form the same judgment of the place of the object as we used to do in the more common cases of direct vision. How we know and judge of the place and position of an object by the place and inverted position of its picture upon the retina, will be shewn in the next chapter; wherein it will appear to be entirely the effect of experience.

102. It is manifest from what has been said, that any point P of an object PQ seen by refractions or reflections, appears somewhere in the line pO, drawn from the corresponding point p of its last image to the eye at any place O. Because all the rays which flowed from P do after the last refraction or reflection flow from or towards the corresponding point p of the last image. The reason why I say the last image will be mentioned in the 111th article.

Apparent directions of visible points determined. Fig. 160 to 172.

103. It is also manifest why an object seen by refracted or reflected rays appears sometimes upright and sometimes inverted. For when the refracted or reflected rays AO, CO, have the same situation with respect to each other, as two rays that come directly from the same points of the object to the eye, these points will appear in the same situation with respect to each other in both cases [b]. But if the rays that come from these

Their apparent situation determined.

b Art. 101.

points shall have crossed each other before they arrive at the eye, they will then have a contrary situation to that of two rays coming directly from the same points to the eye; and consequently these two points will appear in the glass in a contrary situation [a]. And one may add that in the former case, the picture upon the retina of the eye will have the same position, though not the same magnitude, as if the glass was removed, and will have a contrary position in the latter case.

a Art. 101.

Apparent magnitude in glasses defined.

104. The apparent magnitude of an object, *PQ*, seen by refracted or reflected rays either upright or inverted, is a quantity of visible extension, measured by the angle, *AOC*, which the two [b] rays, *AO*, *CO*, that came from its extremities, *P*, *Q*, do make, after their last refraction or reflection, in falling on the eye. Or in other words, the object appears greater or smaller in proportion as that angle *AOC* is greater or smaller. Because its extremities appear in the directions of the last refracted or reflected rays *OA*, *OC* [c]; and also because its picture upon the retina is greater or smaller in proportion as these rays constitute a greater or smaller angle at the eye [d].

b Art. 90.
c Art. 101.
d Art. 91.

And determined.

105. Therefore the apparent magnitude of an object, *PQ*, is also measured by the angle *pOq* which its last image *pq* subtends at the eye. For the lines *AO*, *pO* are but one line continued, and so are *CO*, *qO*, and therefore the angles *AOC*, *pOq* are the same when the image lies before the eye, and are equal when it lies behind it.

How it varies.

106. Hence the apparent magnitude of an object increases and decreases in proportion as the eye approaches to or recedes from its last image, (just as if it was a real object [e],) placed either before or behind the eye. For when the image is fixed, the angle *pOq*, when small, increases in the same proportion as *Oq* decreases, and on the contrary [f].

e Art. 99.
f Art. 60.

When invariable. Fig. 173. to 176.

107. Hence if the last image be removed to an infinite distance, that is, if the object be placed in the principal focus of a lens, sphere, or concave looking-glass, its apparent magnitude to the eye at any place whatever will be invariably the same; and equal to its apparent magnitude seen by the naked eye, supposing it put into the place of the center of the sphere, lens, or reflecting concave. For since all the rays of any one pencil, are parallel to its axis *PE*, the angle *COA*, which measures the apparent magnitude at any point *O*, is every where equal to the angle *QEP* at the center *E*.

Fig. 177, 178.

The apparent magnitude of the object will also be invariable where-ever it be placed, when the eye is fixed at the principal focus of any glass which makes parallel rays converge to the eye. For conceiving them to flow back again from the eye to the object, they will fall upon the same points of the object from whence they came while it is moved in any place along the axis of the glass: and no other rays but these can return from the same points of the object to the eye in that place:

place: therefore the ſeveral parts of the object will always be ſeen under the ſame angles, and conſequently will appear of the ſame magnitudes [a]. a Art. 104.

108. The apparent magnitude of an object ſeen by reflected or refracted rays being meaſured by the angle which its laſt image ſubtends at the eye [b], and its apparent magnitude to the naked eye in any place being meaſured by the angle which the object it ſelf ſubtends at the eye in that place [c], it follows that the former apparent magnitude is to the latter, as the former angle to the latter angle. For the meaſures of things and the things meaſured by them are proportionable.

Compared to the true magnitude.

b Art. 105.

c Art. 97.

109. Conſequently the apparent magnitude of an object ſeen in a glaſs, will be equal to its apparent magnitude to the naked eye in the ſame place, if the glaſs was removed, Firſt, when the object touches any thin lens or any ſingle ſurface. For the image is then equal to the object and coincides with it [d]. Secondly, when the eye touches any thin lens or any reflecting ſurface. For then the ray PAO will paſs from the object to the eye through the middle of the lens very nearly, and therefore being almoſt ſtraight [e] will make nearly the ſame angle with the axis as an unrefracted ray would do: and when the point of incidence, A, coincides with C at any reflecting ſurface, the incident and reflected rays PA, AO, produced, will alſo make equal angles with the axis or perpendicular QC [f]; and ſo the object will appear under the ſame angle as it would do to the naked eye turned about. Thirdly, when the eye is at the center of a reflecting concave. For then the incident and reflected rays PA, AO will coincide with the direct ray PE [g], and conſequently will make the ſame angles with the axis. Fourthly, when the object is at the center of the reflecting concave. For then the reflected image is alſo at the center and is equal to the object [h]. Fifthly, when a ray coming directly from P to O, would make an angle with the axis equal to the angle AOC, which the refracted or reflected ray PAO makes with it on the other ſide.

When equal to the true.

d Art. 55.

e Art. 42.

f Art. 8.

g Art. 10.

h Art. 29. Fig. 165, 167, 171.

110. Theſe caſes being excepted the apparent magnitude of an object ſeen through a concave lens is always leſs than the true; and when it is ſeen upright through a convex lens, or a globe, it is greater than the true. For the ray PAO, coming from the extremity of the object to the eye, is bent by the concave lens from its axis, and therefore makes a leſs angle with it at the eye than a ray coming directly from that extremity to the eye. But the ſame ray is bent by the convex lens towards its axis, and therefore makes a greater angle at the eye than the direct ray: and the apparent magnitudes are meaſured by theſe angles.

Leſs than the true through a concave and greater through a convex glaſs.

111. What has hitherto been demonſtrated concerning the apparent magnitude of an object PQ, will continue in force if you ſuppoſe the object PQ to be an image formed by another glaſs or other glaſſes. For the

The whole applyed to viſion through any number of glaſſes.

the rays diverge from either of them in the same manner, and for this reason I have always called pq the last image of the object.

What part of an object is visible in any glass.

112. The place of the eye at O being given, to determine what part of an object is visible in a given portion or aperture AC of any refracting or reflecting glass, draw OA to the edge of the aperture and produce it till it cuts the image in p, and through the center of the glass draw pE cutting the object in P; and PQ will be the part in view in the aperture AC. For the whole pencil of rays flowing from P will belong to p after refraction or reflection[a], and consequently some one of those rays will advance to the eye in the line AO drawn through p. If the image be at an infinite distance all the rays that belonged to p will be parallel to the axis of the pencil; therefore PQ is now determined by drawing EP parallel to OA. In a plane looking-glass, pP must be drawn from p parallel to qQ, or perpendicular[b] to the glass, to cut off the part PQ visible in the aperture AC. For this glass may be considered as having a center at an infinite distance from it.

a Art. 43.
Fig. 173 to 176.
Fig. 168.
b Art. 25.

How it varies.

113. Hence if the glass and object be fixed, the part in view in a given aperture will decrease perpetually while the eye recedes from the glass; unless the image be behind the eye. For then it will decrease only till the eye arrives at the image, and after the eye has passed by the image it will increase perpetually. The reason is because the object and image, being fixed in their places, do both increase or both decrease together, being both terminated by two lines Pp, Qq that meet or cross in E the center of the glass.

When greatest and least.

114. Therefore the part in view is greatest when the eye is close to the glass, and least when close to the image; and, in this latter case, it appears infinitely magnified. For conceiving the distance Oq infinitely diminished, the parts pq, PQ cut off by the lines AOp, pEP will both be infinitely diminished; but the magnitude of the angle at O, subtended by pq or by AC, continues finite while the angle subtended by PQ at O is infinitely diminished: and so the disproportion between these angles, that is between the apparent and true magnitudes of the particle PQ[c] is infinitely great. It appears also infinitely confused, when the pupil is open, for the reason given in the following articles.

c Art. 108.

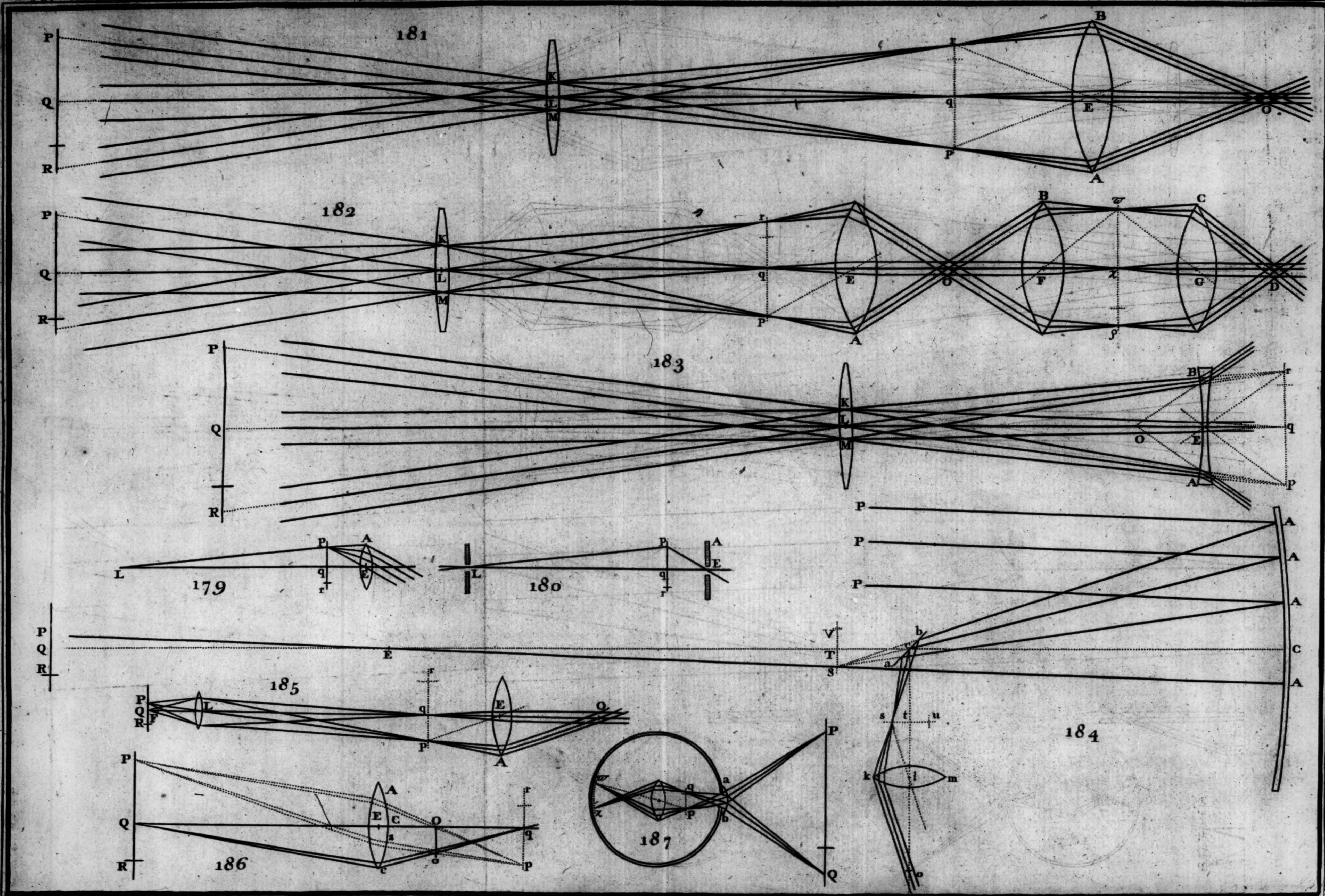
The size of a looking-glass sufficient to see all ones own body.
Fig. 168.
d Art. 23.
e Art. 57.

115. When a person views himself in a plane looking-glass he appears to himself to fill the same part of the glass wherever he stands: and the length and breadth of this part is always half the length and breadth of the corresponding part of his own body. For when O and Q coincide, OC is half of Oq or Qq[d] and therefore AC is half of pq[e] or PQ.

Vision when confused by glasses.
f Art. 90.

116. Hitherto I have considered the pupil of the eye as no bigger than a point, admitting but a single ray from every point of the object[f]; by which means the picture upon the retina would be distinct in all cases.

But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But when the pupil is open, if the image formed by the glaſs be nearer to the eye than the leaſt diſtance at which we can ſee objects diſtinctly with the naked eye, the appearance through the glaſs will be confuſed. Becauſe the rays diverge too much from ſo near an image to be reduced by the eye to a diſtinct picture upon the retina. On the other hand, when the rays converge to an image behind the eye, they will be collected to a diſtinct picture before they arrive at the retina, becauſe the eye is not naturally uſed to conform it ſelf to converging rays; and ſo the viſion will be confuſed in both caſes, but may be rendered diſtinct as followeth.

117. Things which appear confuſed when ſeen by direct, refracted or reflected rays may be rendered diſtinct either by looking through a little hole in a thin plate or bit of paper, or through a convex or concave glaſs of a proper degree of convexity or concavity; and provided the hole or glaſs be put cloſe to the eye, the apparent magnitude and ſituation of the object will be the ſame in both caſes. For if the hole be ſo ſmall as to admit but a ſingle ray from every diſtinct point of the object, theſe rays will fall upon the retina in as many other diſtinct points, and will make a diſtinct picture. And when the pencils of rays fall upon a thin lens, their axes go ſtraight through the middle of it [a], and conſequently will proceed to the ſame points upon the retina as when they paſſed through the hole. Now ſuppoſing the lens to have ſuch a figure that the rays of every pencil ſhall be refracted by it, and by the eye together, to thoſe very points of their axes, which touch the retina, the picture will ſtill be diſtinct: and will be the ſame in magnitude and poſition as before: and the only difference in the effects of the hole and lens will be in the degree of brightneſs of the picture upon the retina.

How rendered diſtinct.

a Art. 43.

118. A ſingle microſcope is only a very ſmall globule of glaſs, or a ſmall double convex glaſs, whoſe focal diſtance is very ſhort. A minute object pq ſeen diſtinctly through a ſmall glaſs AE by the eye put cloſe to it, appears ſo much greater than it would to the naked eye, placed at the leaſt diſtance qL from whence it appears ſufficiently diſtinct, as this latter diſtance qL is greater than the former qE. For having put your eye cloſe to the glaſs EA, in order to ſee as much of the object as poſſible at one view [b], remove the object pq to and fro till it appears moſt diſtinctly, ſuppoſe at the diſtance Eq. Then conceiving the glaſs AE to be removed, and a thin plate, with a pin-hole in it, to be put in its place, the object will appear diſtinct, and as large as before [c], when ſeen through the glaſs, only not ſo bright. And in this latter caſe, it appears ſo much greater than it does to the naked eye, at the diſtance qL, either with the pin-hole or without it, as the angle pEq is greater than the angle pLq [d], or as the latter diſtance qL is greater than the former qE [e].

A ſingle microſcope, how much it magnifies.

Fig. 179, 180.

b Art. 114.

c Art. 117.

d Art. 97.

e Art. 60.

119. Since

And in what manner.

119. Since the interposition of the glass has no other effect than to render the appearance distinct, by helping the eye to increase the refraction of the rays in each pencil, it is plain that the greater apparent magnitude is intirely owing to a nearer view than could be taken by the naked eye. If the eye be so perfect as to see distinctly by pencils of parallel rays falling upon it, the distance Eq, of the object from the glass, is then the focal distance of the glass. Now if the glass be a small round globule whose diameter is $\frac{1}{15}$ of an inch, such as are easily made in the manner described in the third book, its focal distance Eq being three quarters of its diameter, [a] is $\frac{1}{20}$ of an inch; and if qL be 8 inches, the usual distance at which we view minute objects, this globule will magnify at the rate of 8 to $\frac{1}{20}$ or of 160 to 1.

a Art. 61.

Astronomical telescope, how much it magnifies, and why. Fig. 181.

120. An astronomical telescope is composed of two convex glasses in the following manner. PQ represents the semidiameter of a remote object, and pq its picture formed by the convex lens L, which being next to the object is called the object-glass. In the axis of this glass, QLq produced, EA represents another glass more convex than L, so placed, that as qL is the focal distance of the glass L, so qE is the focal distance of the glass E; and EL the sum of their focal distances. In this situation of the glasses, I say the object will appear to the eye at any point O, distinct, inverted and magnified at the rate of qL to qE, that is of the focal distance of the object glass to the focal distance of the eye glass.

For the rays which diverge from the point q of the picture pq, being refracted by the eye-glass, will emerge upon the eye at O in lines parallel to the axis qEO; because qE is supposed to be the focal distance of the eye-glass; and for the same reason, the rays which diverge from any collateral point p, of that picture pq, will emerge from the eye-glass, after refractions at A, in lines parallel to the line or ray pE; this line being the axis of an oblique pencil of rays, part of which diverge from p upon the glass. An eye therefore which can see distinctly by pencils of parallel rays being placed any where at O, among the intersections of these pencils, will see the points of the object distinctly.

Now to the eye at O the apparent magnitude of the picture pq, or object PQ, is measured by the angle EOA [b], or by the equal angle qEp; but to the naked eye at L, if the glass was removed, the apparent magnitude of the object is measured by the angle QLP, or by the equal angle qLp; the oblique axis PLp being straight [c]. Therefore the former apparent magnitude is to the latter, as the angle qEp, to the angle qLp; and consequently as the latter distance qL, to the former qE [d].

b Art. 104.

c Art. 43.

d Art. 60.

A telescope made of four convex glasses considered. Fig. 182.

121. The object which appeared inverted in the former telescope [e], will appear upright and distinct through two more convex eye-glasses subjoined to it; at a distance from each other, equal to the sum of their focal

e Art. 103.

focal distances; and when their focal distances are equal to each other, the object will be magnified just as much as it was before. For the pencils of parallel rays EOF, AOB, &c, which are continued to the glass FB, will be formed by it into a second image $\varpi\kappa$; and the focus ϖ, of any oblique pencil OB, will be determined by the intersection of the line $\varpi\kappa$, perpendicular to the common axis of the glasses, and of the oblique axis $F\varpi$, drawn parallel to the incident rays OB[a]. This point ϖ being the focus of incident rays on the last glass GC, the emergent rays CD will be parallel to their oblique axis ϖG; because the rays that flow from κ are supposed to emerge parallel to the direct axis. Therefore to the eye at D, where these emergent pencils cross, the object will appear distinct, and upright[b]. And when the glasses F and G are exactly equal, the image $\varpi\kappa$ will be exactly in the middle between them; and so the triangles $\varpi F\kappa$, $\varpi G\kappa$ will be exactly equal. Consequently the angle CDG, which now measures the apparent magnitude to the eye at D, will be equal to the angle $\varpi G\kappa$ or $\varpi F\kappa$ or BOF or AOE, which measured it before to the eye at O.

a Art. 55.

b Art. 103.

How much they take in at one view. Fig. 181, 182.

122. In a telescope of a given length the quantity of objects taken in at one view, depends upon the breadth of the eye-glass. For as AE is greater or smaller, the angle ALE or its equal PLQ is also greater or smaller; and this angle takes in all the objects that can be seen at one view on one side of the axis of the telescope.

Galileo's telescope considered. Fig. 183.

123. The difference between the astronomical telescope and *Galileo*'s telescope or a common perspective-glass is this; instead of the convex eye-glass placed behind the image to make the rays of each pencil go parallel to the eye, there is placed a concave eye-glass AE as much before it; which opens the rays of each pencil that converged to q and p, and makes them emerge parallel upon the eye; as is evident by conceiving the rays to go back again through the eye-glass, whose focal distance we supposed was Eq. The eye must be put close to the glass to receive as many pencils as possible; and then, supposing an emerging ray of an oblique pencil produced backward along AO, the apparent magnitude of the object is measured by the angle AOE[c] or its equal qEp, which is to the angle qLp (or QLP, the measure of the true magnitude,) as qL to qE, as before in the other telescope. It is manifest, by the 103d article, that objects in this telescope appear upright.

c Art. 104.

This takes in less than the former.

124. The quantity of objects taken in at one view in this telescope does not depend upon the breadth of the eye-glass, as in the astronomical telescope, but upon the breadth of the pupil of the eye. Because the pupil is less than the eye-glass, and the lateral pencils do not now converge to but diverge from the axis of the glasses. Upon this account the view being narrower is not so pleasant as in the former telescope.

125. Sir

Sir *Is. Newton*'s reflecting telescope. Fig. 184.

125. Sir *Isaac Newton*'s reflecting telescope magnifies the diameter of a remote object in the proportion of the focal distance of the reflecting concave to the focal distance of the convex eye-glass and shews it inverted. Let *ST* be an image of a remote object *PQ* formed by reflections from a large concave surface *AC*, and terminated by the lines *PESA*, *QETC* drawn through its center *E*. Now because this image cannot be viewed through an eye-glass placed directly before it (for then the spectator would intercept the rays that are coming to the concave) therefore let the several pencils of rays which converge towards it in coming from the broad concave *AC*, be reflected sideways from a small polished plane, represented by *ac*; and then the second image *st*, formed by this plane, will be equal to the first image *ST*[a]. Let *tl* be the focal distance of a small convex eye-glass *kl* and the rays which flow from any point *s* will be refracted through this glass, to the eye at *o*, in the lines *ko* drawn parallel to the oblique axis *sl*; and so the apparent magnitude of the object, *PQ*, to the eye at *o*, will be measured by the angle *kol* or *slt*[b]: but to the naked eye at *E*, it is measured by the angle *PEQ* or *SET*. Therefore the former apparent magnitude is to the latter, as the angle *slt* to the angle *SET* or, (because their subtenses *st*, *ST* are equal,) as *ET* to *lt*[c] or as *CT* to *lt*, when the object is remote[d]. Note that the plane *acb* is much too broad in comparison to the concave *ACB*, which could not be helped in so small a draught. That the appearance of the object is inverted or turned from right to left, is evident by the 103d article.

a Art. 24, 25.
b Art. 104.
c Art. 60.
d Art. 26.

Why so much shorter than others.

126. Dioptrick telescopes which magnify much being very long and troublesome to be managed, Sir *Isaac Newton* proposed this method to shorten telescopes [1]: which answers to admiration; as appears by a table in the next book, of the lengths of both sorts of telescopes which magnify equally with equal distinctness. The reason why dioptrick telescopes cannot be shortened as much as these, and still magnify as much, by diminishing the focal distances of the eye-glasses[e], in short is this. The images made by refractions through the convex object-glasses, being much more imperfect than those, which are made by reflections from concave surfaces, will not bear to be magnified so much by so small eye-glasses[f], without appearing confused: and the chief cause of those imperfections in the pictures is the unequal refrangibility of rays of different colours; as will be fully explained hereafter.

e Art 120.
f Art. 118.

Double microscope considered. Fig. 185.

127. A double microscope is composed of two convex glasses placed at *E* and *L*. The glass *L* next the object *PQ* is very small and very much convex, and consequently its focal distance *LF* is very short; the distance *LQ* of the small object *PQ* is but a little greater than *LF*; so that the image *pq* may be formed at a great distance from the glass[g],

g Art. 48.

1 Opticks. p. 95.

and

and consequently may be much greater than the object it self [a]. This picture pq being viewed through a convex eye-glass AE, whose focal distance is qE, appears distinct as in a telescope. Now the object appears magnified upon two accounts; first because if we viewed its picture pq, with the naked eye, it would appear as much greater than the object, at the same distance, as it really is greater than the object, or as much as Lq is greater than LQ [b]; and secondly because this picture appears magnified through the eye glass as much as the least distance at which it can be seen distinctly with the naked eye, is greater than qE, the focal distance of the eye-glass [c]. For example, if this latter ratio be 5 to 1, and the former ratio of Lq to LQ be 20 to 1, then upon both accounts the object will appear 5 times 20, or 100 times greater than to the naked eye.

[a] Art. 55.
[b] Art. 55.
[c] Art. 118.

To fit telescopes and microscopes to defective eyes.

128. To fit these telescopes and microscopes to short-sighted eyes, the glasses E and L must be placed a little nearer together; so that the rays of each pencil may not emerge parallel but may fall diverging upon the eye [d]; and then the apparent magnitude will be altered a little but scarce sensibly.

[d] Art. 48.

The apparent brightness through them.

129. The brightness of the appearance through a given telescope or microscope is more or less in proportion to the aperture of the object-glass. For supposing it covered with paper, all but a small hole in the middle, the magnitudes of the pictures pq in the focus of the glasses, and of that upon the retina would not be altered; but the hole at L being smaller than before, there are fewer rays in every pencil, and consequently in every point of those pictures, and so they appear more obscure. If the aperture and object-glass remain the same, things appear brighter or fainter according as the focal distance of the eye-glass is longer or shorter; that is, according as the telescope or microscope magnifies less or more [e]. For the same quantity of light spread over a smaller or larger picture or part of the retina will make it brighter or duller.

[e] Art. 126, 127.

All appearances the same when the eye is out of the axis of the glass or glasses. Fig. 186.

130. Hitherto I have supposed the eye to be always placed at some point O in the common axis of the refracting or reflecting surfaces. Now let it be placed at any point o in a line Oo perpendicular to the axis Qq; I say that all the appearances will be the same or at least not sensibly different from what they were before. For let pq be the last image of an object, and PQ the last but one, or the object it self; draw the lines po, qo meeting the next surface in a and c; and the points P and Q will appear to the eye at o in the directions of those lines oa, oc. Whence drawing pO meeting the surface in A; since the directions OA, oa, in which P is seen, lye the same way from the directions OC, oc, in which Q is seen, it is evident that the apparent situation of the extremities P, Q is the same at both places of the eye; and also the apparent magnitude, which is measured by the angle aoc [f] or poq or pOq or AOC. For the small

[f] Art. 104.

small angles poq, pOq, being subtended by the same image pq, very nearly at equal distances po, qO from o and O, are very nearly equal. The apparent brightness of the object is also the same; because the density of the rays, that enter the pupil, at any part of the perpendicular plane represented by Oo, is nearly the same [a]. For the rays flow from or towards the last image pq just as if it was a luminous body. And lastly the degree of apparent distinctness or confusion is the same also, because the angles which the pupil, placed at O or at o, subtends at p and q, or the mutual inclinations of the rays in each pencil are very nearly equal.

a Art. 58.

A general observation upon vision.

131. This general observation upon vision is worth remembring. That the apparent distinctness and confusion of an object depends upon the mutual inclination of the rays to each other in any one pencil when they fall upon the eye [b]; the apparent magnitude, upon the inclination of the rays of different pencils to each other when they fall upon the eye [c]; the apparent situation, upon the real situation of the extream pencils when they fall upon the eye [d]; and the apparent brightness and obscurity, upon the quantity of rays in every pencil [e].

b Art. 116.

c Art. 104.

d Art. 103.

e Art. 68.

CHAP. V.

CONCERNING OUR IDEAS ACQUIRED BY SIGHT.

The case of persons born blind considered.

132. IN order to account for several appearances in vision, it is necessary to consider the manner of acquiring our ideas of things by sight. The noted question proposed by Mr. *Molyneux* to Mr. *Locke*, whether a person blind from his birth, being made to see, could by sight alone distinguish a globe from a cube, whose difference he knew by feeling, has been pronounced in the negative by both those philosophers [1]: and this opinion has since been confirmed by the experience of several persons, who receiving their sight from the operation of Couching, could not know any one thing from another, however different in shape and magnitude. Mr. *Chesselden* having given us a very curious account of some observations made by a young gentleman who was couched by him in the thirteenth year of his age, I will here insert it in his own words [2].

Mr. *Chesselden*'s account of such persons brought to sight.

133. Though we say of this gentleman that he was blind, as we do of all people who have ripe Cataracts, yet they are never so blind from that cause, but that they can discern day from night; and for the most part in a strong light, distinguish black, white, and scarlet, but they cannot perceive the shape of any thing: for the light by which these perceptions are made, being let in obliquely through the aqueous humour,

1 *Locke*'s Essay on Hum. Underst. B. 1. c. 9.
2 Phil. Trans. No. 402. See another account in the 55th Tatler.

or

or the anterior ſurface of the cryſtalline (by which the rays cannot be brought into a focus upon the retina) they can diſcern in no other manner, than a ſound eye can through a glaſs of broken jelly where a great variety of ſurfaces ſo differently refract the light, that the ſeveral diſtinct pencils of rays cannot be collected by the eye into their proper foci; wherefore the ſhape of an object in ſuch a caſe, cannot be at all diſcerned, though the colour may: and thus it was with this young gentleman, who though he knew theſe colours aſunder in a good light; yet when he ſaw them after he was couched, the faint ideas he had of them before, were not ſufficient for him to know them by afterwards; and therefore he did not think them the ſame, which he had before known by thoſe names. Now ſcarlet he thought the moſt beautiful of all colours, and of others the moſt gay were the moſt pleaſing; whereas the firſt time he ſaw black, it gave him great uneaſineſs, yet after a little time he was reconciled to it: but ſome months after, ſeeing by accident a Negro woman, he was ſtruck with great horror at the ſight.

When he firſt ſaw, he was ſo far from making any judgment about diſtances that he thought all objects whatever touched his eyes (as he expreſſed it) as what he felt did his ſkin; and thought no objects ſo agreeable as thoſe which were ſmooth and regular, though he could form no judgment of their ſhape, or gueſs what it was in any object that was pleaſing to him. He knew not the ſhape of any thing, nor any one thing from another, however different in ſhape, or magnitude, but upon being told what things were, whoſe form he before knew from feeling, he would carefully obſerve, that he might know them again; but having too many objects to learn at once, he forgot many of them: and (as he ſaid) at firſt he learned to know, and again forgot a thouſand things in a day. One particular only (though it may appear trifling) I will relate: Having often forgot which was the cat, and which the dog, he was aſhamed to aſk, but catching the cat (which he knew by feeling) he was obſerved to look at her ſtedfaſtly, and then ſetting her down, ſaid, ſo puſs, I ſhall know you another time. He was very much ſurprized, that thoſe things which he had liked beſt, did not appear moſt agreeable to his eyes, expecting thoſe perſons would appear moſt beautiful that he loved moſt, and ſuch things to be moſt agreeable to his ſight that were ſo to his taſte. We thought he ſoon knew what pictures repreſented, which were ſhewed to him, but we found afterwards we were miſtaken: for about two months after he was couched he diſcovered at once, they repreſented ſolid bodies; when to that time he conſidered them only as party-coloured planes, or ſurfaces diverſified with variety of paint; but even then he was no leſs ſurprized, expecting the pictures would feel like the things they repreſented, and was amazed when he found thoſe parts, which by their light and ſhadow appeared now round and uneven,

felt only flat like the rest: and asked which was the lying sense, feeling or seeing?

Being shewn his father's picture in a locket at his mother's watch, and told what it was, he acknowledged a likeness, but was vastly surprized; asking how it could be, that a large face could be expressed in so little room; saying, it should have seemed as impossible to him, as to put a bushel of any thing into a pint.

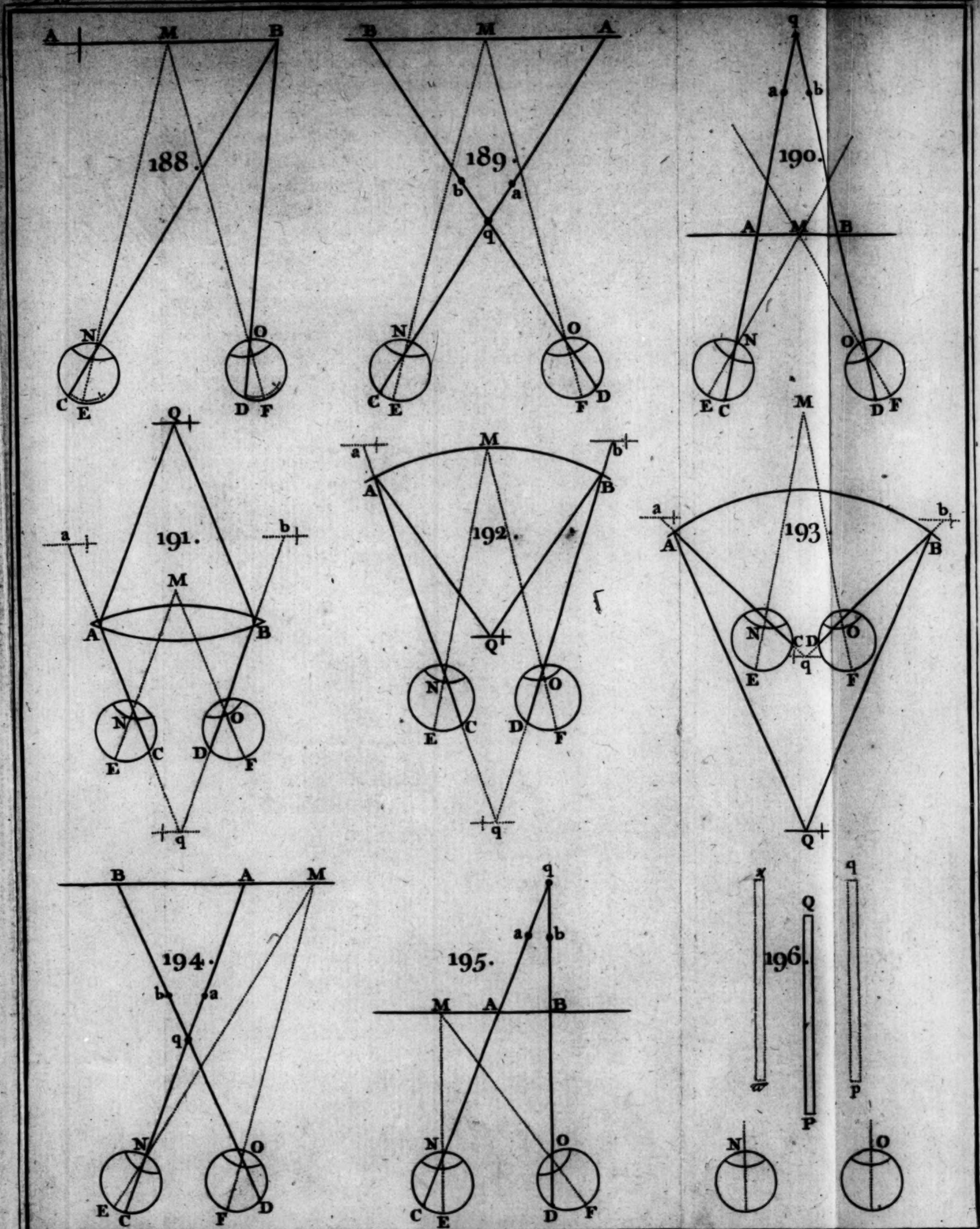
At first he could bear but very little light, and the things he saw, he thought extreamly large; but upon seeing things larger, those first seen he conceived less, never being able to imagine any lines beyond the bounds he saw: the room he was in, he said, he knew to be but part of the house, yet he could not conceive that the whole house could look bigger. Before he was couched, he expected little advantage from seeing, worth undergoing an operation for, except reading and writing; for he said, he thought he could have no more pleasure in walking abroad than he had in the garden, which he could do safely and readily. And even blindness he observed, had this advantage, that he could go any where in the dark much better than those who can see; and after he had seen, he did not soon lose this quality, nor desire a light to go about the house in the night. He said every new object was a new delight, and the pleasure was so great, that he wanted ways to express it; but his gratitude to his operator he could not conceal, never seeing him for some time without tears of joy in his eyes, and other marks of affection: and if he did not happen to come at any time when he was expected, he would be so grieved, that he could not forbear crying at the disappointment. A year after his first seeing, being carried upon *Epsom* Downs, and observing a large prospect, he was exceedingly delighted with it, and called it a new kind of seeing. And now being lately couched of his other eye, he says, that objects at first appeared large to this eye, but not so large as they did at first to the other: and looking upon the same object with both eyes, he thought it looked about twice as large as with the first couched eye only, but not double, that we can any ways discover.

Some additions to that account.

134. Mr. *Chesselden* adds in another paper printed by it self, that he has brought to sight several others who had no remembrance of ever having seen; and that they all gave the same account of their learning to see, as they called it, with the young gentleman above mentioned, though not in so many particulars; and that they all had this in common, that having never had occasion to move their eyes they knew not how to do it; and at first, could not at all direct them to a particular object; but in time they acquired that faculty, though by slow degrees.

By what regular steps they may learn to know things.

135. Let us now consider a little by what regular steps and observations a person in this case, might learn to know the places, magnitudes, figures,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figures and distances of objects. Since he cannot direct his eye to look at any particular object[a], whose place he knows by feeling, at first we must suppose his eye at rest; and when he has learned to know his hand, or his finger end, let him move it gently upwards and downwards. During this motion he cannot help perceiving some sort of alteration in the visible appearance, occasioned by the corresponding motion of the picture of his finger over different parts of the retina. Then by carefully observing and remembring what sort of sensation was perceived when his finger was in any particular place, suppose above his eye; whenever the like sensation shall again be excited, by another picture of the same or of a different object, falling upon the same place of the retina, wherever it be, he will conclude that this object, whose place is unknown, is above his eye, or in the place where he formerly held his finger. By the like observations made with his hand and frequently repeated, he may learn to know the motion of a body by sight; and the direction of its motion, with respect to his own body; and consequently to know extension and the situation of extension; and consequently to know the figure of bodies, which consists only of various extensions variously situated. This he may learn by moving his finger round the edges of bodies, and by observing the various bendings of its visible motion; or by walking round a room; and in general by comparing together the ideas suggested to his mind by sight and by touch. And by observing that the appearance of the same body is continually varied while the eye approaches to it or recedes from it, by this variety of apparent magnitude he will learn to know the distances of things from his eye as well as from one another. Lastly since he cannot help perceiving those objects most distinctly which lye nearest to the axis of the eye produced, and others more confusedly which lye farther and farther from it, as other people do; when he finds that an object distinctly perceived, does suddenly grow confused, by any accidental motion of his head or eye, the memory of that distincter perception now lost will induce him to try to regain it, by a voluntary motion of his head or eye; till by frequent trials he will learn to direct his eye to any desired object. And by the same method he will learn to direct both eyes to the same object. From what has been said it appears that our perception of things by sight is no more than this: by memory of former perceptions by sight and other senses compared together, we collect in an instant that the thing we now perceive by sight only will affect our other senses, upon trial, as it formerly used to do. I say in an instant, which will less surprize us, when we consider how quick the characters or sounds of words, whose signification we could hardly remember at first, do excite in our minds the ideas of things they are constantly used to signify: so great is the force of habits in bringing our ideas together. And so it appears at

a Art. 134.

at last, that the manner, wherein external objects are signified to us, by the sensations of light and colours, is the same with that of languages and signs of human appointment: which do not suggest the things signified by any likeness or identity of nature, but only by an habitual connection that constant experience has made us observe between them *.

The effect of inverted pictures on the retina considered.

136. Now if it be the memory of the same sensations excited in the same places of the retina, though unknown, which occasions the same judgment of the place of an object, (which will be farther confirmed in the following articles,) the inverted pictures on the retina will serve as well to excite the same ideas as if they had been constantly upright or in any oblique posture. It is only necessary that the object and picture should alter their places both together by any constant rule whatever. For example, if a person was born with an eye in which the pictures of objects were painted upright, as represented in the 187th figure, where the first image pq is formed by the cornea in the aqueous humour enlarged, and the second image by the crystalline upon the retina $\varpi\kappa$; let it be considered whether by these upright pictures he might not learn to know, judge, and talk of objects just as other people do, and on the contrary; and whether it could easily be known that his eyes were different from common eyes.

Vision with both eyes when single when double. Fig. 188.

137. The axis of the eye is a line drawn through the middle of the pupil and of the crystalline, and consequently falls upon the middle of the retina. And the axes of both eyes produced are called the optick axes. When the optick axes are parallel or meet in a point, the two middle points of the retinas, or any two points which are equally distant from them, and lye on the same sides of them either towards the right hand or left hand, or upwards or downwards, or in any oblique direction, are called corresponding points. Now we find by experience that an object or point of an object appears single when its pictures fall upon corresponding points of the retinas, and double when they do not. For when we view an object steadily, we have acquired a habit of directing the optick axes to the point in view; because its pictures falling upon the middle points of the retinas are then distincter than if they fell upon any other places; and since the pictures of the whole object are equal to one another and are both inverted with respect to the optick axes, it follows that the pictures of any collateral point of the object are painted upon corresponding points of the retinas. This habit of directing the optick axes to the point in view, is so strong that it is very difficult to do otherwise; insomuch that when one eye is shut and the other is in motion, one may feel by ones fingers laid upon the eye-lid, that the eye which is shut, always follows the motions of the eye that is open. But if by squinting or by depressing an eye with ones finger, the

* See *Berkeley's* Essay on Vision. p. 172.

optick

optick axes are not directed to the same point; in these cases objects appear double: and now it is plain that their pictures are not painted upon corresponding places of the retinas.

For the like reason if while the optick axes NM, OM are directed to a mark M, we attend to an object or image q, placed any where within the angle NMO or its opposite, made by the optick axes produced, the object q will appear in two places, suppose at a and b, situated in the directions of the visual rays Nq, Oq [a]. For the pictures of the object, q, which lyes between the optick axes, being both inverted with respect to the axes, must fall upon the retinas on contrary sides of the axes, and consequently upon places that do not correspond. And this may be the reason of the double appearance. Because this situation of the pictures never happens in the ordinary and constant use of our eyes, but from two objects A and B placed on opposite sides of the mark M. One of these objects alone being on the same sides of both axes, would have its pictures on corresponding points of the retinas and consequently would appear single. Add to this, that at one view we generally regard no other objects but those which lye round about the mark M, much at equal distances from the eyes; and not those which lye in a long line extended from us. For these objects being placed at various distances from the eye cannot be seen distinctly all at once: it being necessary for this purpose to alter both the distances of the point of concourse of the optick axes, and also the configuration of the eye; that the pictures formed by rays coming from different distances, may be successively distinct. Fig. 189, 190. a Art. 101.

In like manner if the image q of an object Q be formed at any place behind the eyes, either by reflected or refracted rays; and the glass AB be sufficiently broad to throw the rays into both eyes; the object Q will always appear double. For to procure distinct vision we are accustomed to direct the optick axes to some point M lying before them. But the visual rays, ANq, BOq, by which the object is seen, are tending to meet at q behind the eyes, and consequently must fall at C and D on the insides of their axes, which are not corresponding places. Fig. 191, 192, 193.

I find by experience that the apparent distance between the two apparent places of the object, is nearly proportionable to the sum of the arches, CE, DF, upon the retinas; or to the sum of the angles aNM, bOM made by each optick axis and each visual ray, provided these arches lye both on the insides or both on the outsides of the optick axes: but if one lyes within and the other without, the apparent distance of the places of the object is measured by the differences of those arches. For though I have hitherto supposed the object to be within the angles made by the optick axes, in order to shew the effect of the double appearance more plainly, yet it is still double though the object be placed in either axis, or on the outsides of both either much nearer or remoter than Fig. 194, 195.

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their point of concourſe. I find alſo that in all ſituations, the apparent interval, between the two apparent places, will continue the ſame while the eyes are rolled about, ſo as to view any objects placed nearly at equal diſtances from them: and that each image a or b appears over againſt the ſame object A or B when both eyes are open, as it does when the other eye is ſhut. And that when the object, or image q formed by the glaſs, is between the eyes and the mark we look at, the apparent image on the right hand is perceived by the left eye, and the apparent image on the left hand by the right eye; as is manifeſt by opening and ſhutting each eye by turns: but when the object it ſelf, or its image formed by the glaſs, is beyond the mark, or behind the eyes, the apparent image on the right hand is perceived by the right eye, and that on the left hand by the left eye.

Fig. 189.

Fig. 190.

Hence it is manifeſt that the two apparent places a, b of the object q are neither of them the ſame as its real place; and that they lye between it and the mark we look at, but not very far from the real place.

Fig. 196. A double appearance will alſo be ſeen when the end of a long ruler is placed between the eye brows and extended directly forward with its flat ſides reſpecting the right hand and the left; and by directing the eye to a remote object, the right ſide of the ruler, ſeen by the right eye, will appear on the left hand, and the left ſide on the right hand; as repreſented in the 196th figure; in which PQ denotes the ruler, pq and $\varpi\kappa$ its images ſeen by the eyes N, O reſpectively.

Now if it be aſked why in ſeeing with both eyes we do not always ſee double, becauſe of a double ſenſation; I think it is ſufficient to ſay that in the ordinary uſe of our eyes, in which the pictures of an object are conſtantly painted upon correſponding places of the retinas, the predominant ſenſe of feeling has originally and conſtantly informed us that the object is ſingle. By this means our idea of its outward place is connected with both thoſe ſenſations, as is manifeſt by its appearing in two places when its pictures are not painted upon correſponding places of the retinas in the extraordinary circumſtances above mentioned; which is only a direct conſequence ariſing from our general habit of ſeeing. Beſides, whatever anſwer is ſufficient to this queſtion, muſt equally ſerve by the rules of philoſophy, for an anſwer to all others of the ſame ſort: as how it happens that in hearing with two ears we do not hear double; that in feeling with two feet or two hands or two fingers, we do not feel double; as we really do in the dark, when a button is preſſed with two oppoſite ſides of two contiguous fingers laid acroſs; for this reaſon, that thoſe oppoſite ſides of the fingers have never been uſed to feel one but always two things at a time. We have learned therefore by experience of both ſenſes compared together, to make their informations conſiſt-

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ent with each other. Mr. *Cheſſelden* mentions the caſe of a gentleman, who having had one of his eyes diſtorted by a blow on his head, found that every object appeared double; but by degrees the moſt familiar ones became ſingle, and in time all objects became ſo, though the diſtortion ſtill continued *: which greatly confirms the preſent argument, that the judgments we make of the number and places of external objects are entirely the effect of experience; by which our ideas of their number and places are conſtantly connected with certain ſenſations in correſponding places of the retinas: inſomuch that if an animal had a hundred eyes, in the ordinary and conſtant uſe of them an object would appear to him ſingle, and centuple in caſes extraordinary, like thoſe above mentioned.

138. The apparent diſtance of an object, perceived by ſight, is an idea of a real diſtance uſually meaſured by feeling, as by the motion of the body in walking, or otherwiſe; and is ſuggeſted to the mind by the apparent magnitude of the object in view, if ſeen alone, (as a bird in the air, or as an object in a teleſcope or microſcope;) but if it be ſeen with other objects, as it uſually happens, its diſtance is ſuggeſted both by its own apparent magnitude and by the apparent magnitudes of other adjoining objects obliquely extended between the eye and the object in view; as the ſurface of the ground, rivers, walks, high-ways, hedges and ditches, or the houſes in a ſtreet, or the walls and ceiling of a room, or the ſky over head. For what is the apparent magnitude or apparent extenſion of an object but the apparent diſtance of its extremities from one another? and what is the apparent diſtance between two objects in any ſituation or between one object and the ſpectator himſelf, but the apparent extenſion of intermediate objects? And ſince they are ſeldom ſeen alone, excepting through glaſſes, it cannot be doubted, but we eſtimate their diſtances from one another and from our ſelves by our ideas of the magnitudes of thoſe intermediate objects: and every one knows that ſurveyers, gunners, travellers and all ſorts of artificers, who are converſant in meaſuring diſtances, are able to make a truer eſtimate of diſtance by the eye, than others that have not had ſo much experience. Sometimes indeed without attending to thoſe oblique ſurfaces we are ſenſible of the approach of a body by the increaſe of its own apparent magnitude, and on the contrary; and ſometimes we are alſo ſenſible of it when the body is at reſt, provided it be known and familiar to us. For bodies are diſtinguiſhed into ſorts chiefly by their ſhapes and colours, and we reckon them ſmall or great, not in compariſon with bodies of another ſort but with one another; and having found by experience that certain quantities of apparent magnitude of a known body are conſtantly attended by certain quantities of diſtance; the ſenſation of the magnitude of the

Apparent diſtance what and how perceived.

* Anatomy p. 324 3d Edit.

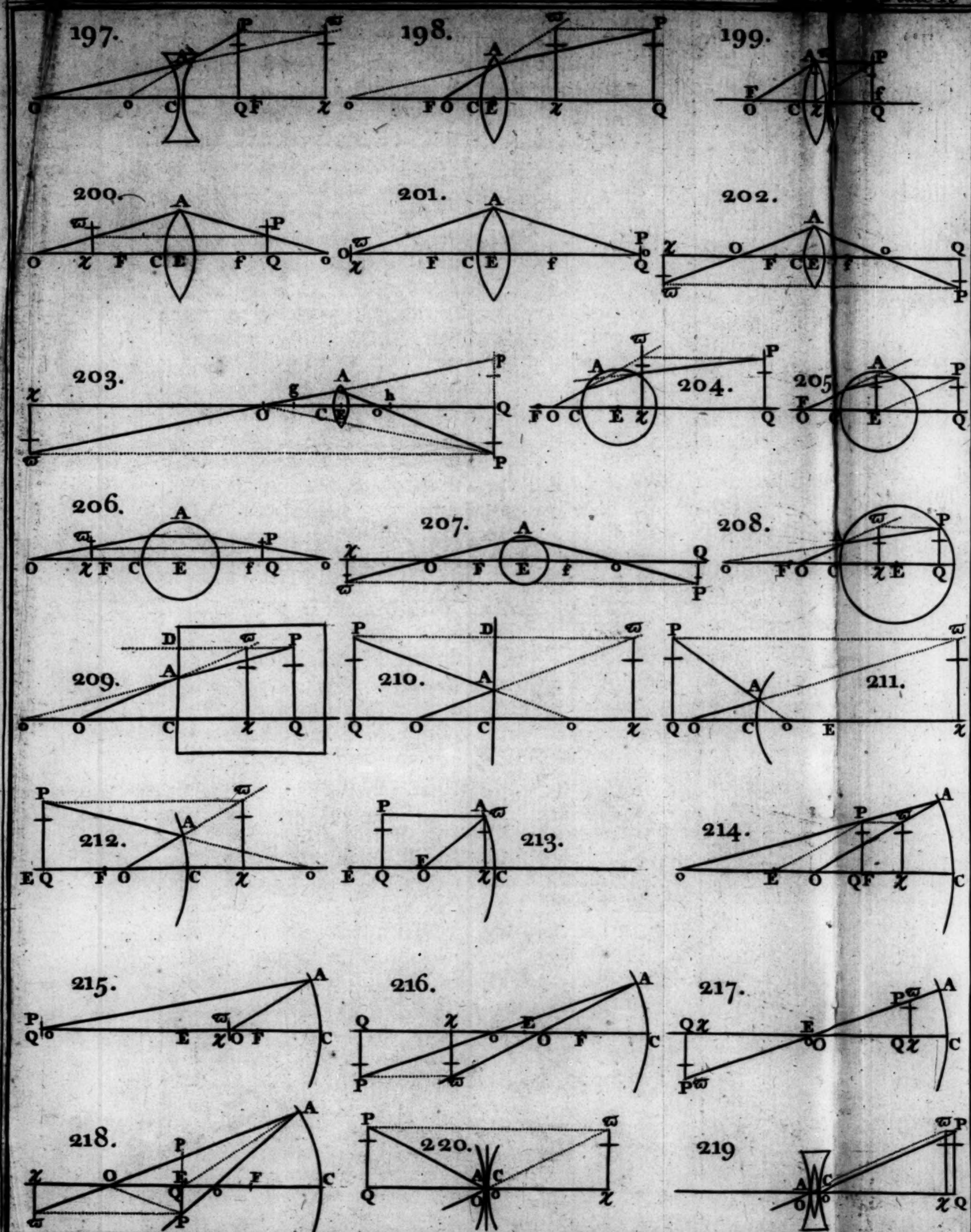
G body

body immediately excites the usual idea of its distance: which is also evident in oblique surfaces as well as of those that are perpendicular to the eye. For the ideas of variable distances must either mediately or immediately be excited in the mind by certain variable sensations, caused by some certain variations in the pictures upon the retina. But while the distance of the object varies, nothing is varied in its picture, excepting its magnitude; its figure, colour, brightness [a] and distinctness receive no sensible variation in most cases: and for one idea to excite another, every one knows it is sufficient that they have constantly been observed to go together, as in languages and a thousand things besides. Lastly I have found by abundance of experiments made with glasses of all sorts, that while the apparent magnitude of an object increases by moving the glass, eye or object, it always appears to approach, and to recede while its apparent magnitude decreases, excepting a particular case or two to be mentioned hereafter. And these experiments seem to me to put the question beyond dispute. For in looking through glasses with one eye only, and at a single object, when nothing is perceived in the space interposed, how is it possible for different apparent magnitudes of the object to suggest the ideas of different quantities of that invisible space, according to a certain rule to be mentioned hereafter; if those ideas had not usually gone together before we looked into the glasses? I find also that by altering the degrees of apparent brightness and distinctness of an object, either by looking through little holes made with a pin or through lenses of different figures put close to my eye, or through both at once put close together and to my eye, that neither the apparent magnitude [b], nor apparent distance is sensibly altered thereby. The reason is, we have had no experience in such confused vision with the naked eye, and therefore, though different degrees of confusion and distinctness in glasses are plainly perceived, yet like the words of an unknown language, their signification of distance or of any thing else is entirely unknown. The same may be said of the degrees of brightness and obscurity. By daylight objects appear equally bright at all moderate distances from the eye [c], and we retain much the same ideas of their distances in the night, when we see them more obscurely. The permanent colours and shades of bodies serve chiefly to distinguish their apparent shapes; and their colours and shapes are manifest distinctions of their various sorts, but being permanent they are no distinctions of their apparent distances from the eye. When the eye is fixed and a fixt line is extended from it, the divergency of rays from different points of that line is neither distinguished nor so much as perceived by sense, by persons that see distinctly. It is a rational deduction from sense which informs us that rays diverge from the points of an object; which the majority of mankind are entirely ignorant of: and the ancient philosophers who thought that something like

a Art. 93.

b Art. 109, 117.

c Art. 93.



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like rays proceeded from the eye to the object, could distinguish distances as well as we. Therefore the divergency of rays from points at different distances is not the medium which introduces the ideas of distances into the mind. Sometimes indeed there are degrees of distinctness and confusion consequent upon it, but their relation to distance as I said before is not perceived. Besides this, in vision with glasses, we have ideas of as many different degrees of distance conveyed to us, as well when the rays come converging towards points behind the eye as when they diverge from points before it, as will be shewn hereafter. The divergency of rays from the place of an object is therefore no cause of its appearing in that place. It is also matter of fact in painting and perspective, that our sensible ideas of the places of the objects in the picture are quite different from our rational ideas of the places from whence the rays diverge: and the difference in these ideas is caused by the different apparent magnitudes of the known objects represented in the picture. It is also evident that our sensible ideas of the places of the remoter parts of a long walk or gallery, and of the clouds over head, and of all celestial bodies, are quite different from the rational ideas of the places from whence the rays diverge, as will appear more fully hereafter. Neither is distance suggested to the mind by the magnitudes of the angles in a triangle made by the optick axes and the interval between the eyes. For these angles are all varied by turning the head sideways while we look at an object, till at last we see it at the same distance with one eye as with both: which shews also that the faint and confused appearance of collateral objects does not alter our ideas of their distances. Nor is distance suggested by feeling the turn of the eyes in widening or contracting the interval between pupils, when we direct them to different places. For the place of the object is generally perceived by a side-view, before we direct our eyes to view it more distinctly. From what has been said it appears to me that the ideas of distance are suggested to the mind by the ideas of the magnitudes of objects.

139. Hence it follows that an object seen by refraction or reflection, appears at the same distance from the eye, as it usually does from the naked eye, when it appears of the same magnitude as in the glasses. To determine this distance in all cases, I conceive a ray OA to go from the eye at O, and after its last reflection or refraction to belong to the focus o, in the common axis OCQ of all the surfaces; and to meet an object PQ in P, placed perpendicular to OQ; and that a line $P\varpi$ is drawn parallel to the axis OQ till it meets the ray OA, produced, in ϖ. Then supposing the object PQ to be removed to the place ϖx, and there to be viewed by the naked eye; since it appears under the same angle ϖOx or AOC as it appeared under in the glasses, when it was at PQ, it will also

Apparent distance in glasses determined. Fig. 197 to 226.

also appear of the same magnitude and consequently at the same distance from the eye in both cases [a]. Therefore if when the object is placed at $\varpi\gamma$, its apparent distance from the naked eye be represented by its real distance $O\varkappa$, the same $O\varkappa$ will also represent its apparent distance in the glasses when it was at PQ. I shall therefore call $O\varkappa$ the apparent distance of the object PQ, and $\varpi\varkappa$ the apparent object.

a Art. 138.

When the point P and the ray OA, by which it is seen, are on contrary sides of the axis OQ, the point ϖ and the line $\varpi\varkappa$ will be behind the eye, and therefore must be viewed by the naked eye inverted and turned about. But if you had rather $\varpi\varkappa$ should always be before the eye, in this case invert the object PQ, and then slide it along the axis; and its extremity P will touch the visual ray OA, produced, at the same distance from the eye as before; because the opposite angles AOC, $\varpi O\varkappa$ are equal.

It varies reciprocally as the apparent magnitude varies.

140. Hence while the eye, object or glasses are in motion, the apparent distance of the object will increase in the same proportion as its apparent magnitude decreases; and on the contrary. For the same apparent distance of the same object seen at $\varpi\varkappa$ by the naked eye, varies in that proportion of the same apparent magnitude [b].

b Art. 99.

Apparent and true distance compared in general.

141. Hence also the apparent distance, $O\varkappa$, of an object, PQ, seen in glasses, is to its apparent distance, OQ, seen by the naked eye, as its apparent magnitude to the naked eye, to its apparent magnitude in the glasses. For conceiving a line OP, which is omitted in the figures for the sake of simplicity; since PQ and $\varpi\varkappa$ are equal, the former distance $O\varkappa$, is to the latter OQ, as the latter angle POQ, to the former $\varpi O\varkappa$ [c].

c Art. 60.

Design.

The ratio of the apparent and true magnitudes of objects being determined in most cases in the foregoing chapter, their apparent distances are also determined by this rule. But because this subject of apparent distance, has hitherto been handled but very imperfectly by all optick writers, it may not be unacceptable to some readers to see it pursued a little farther. I will therefore deduce all the cases of apparent distance immediately from the definition of it [d], without the help of those former demonstrations.

d Art. 139.

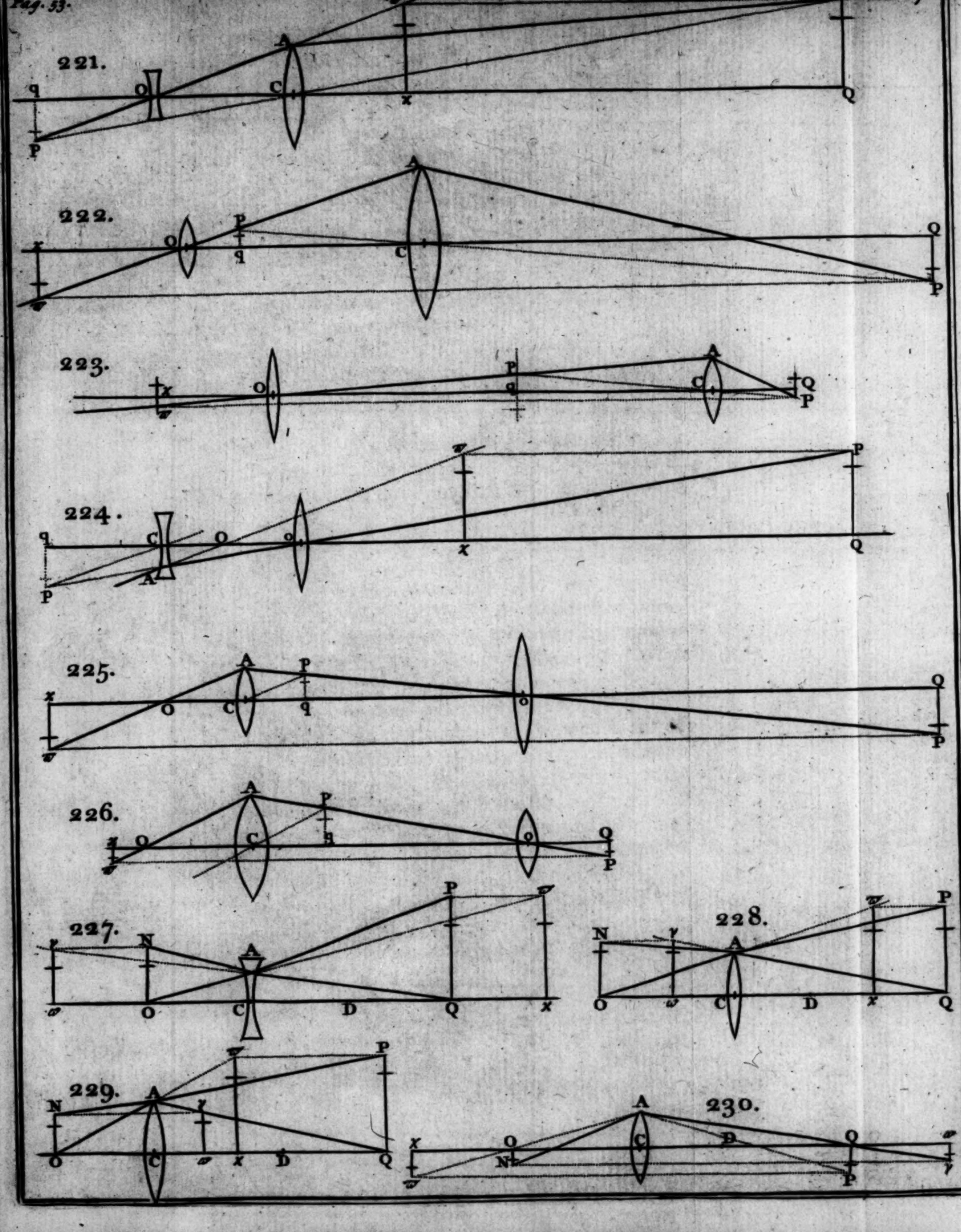
Cases when equal to each other.

142. The apparent and true distances $O\varkappa$ and OQ will be equal, first when the object touches any thin lens or any single surface. For then the points P, A, ϖ will coincide. Secondly, when the eye touches any thin lens or reflecting surface. For when the points O, A, C coincide at a lens, the visual ray will pass through the middle of it very nearly; and consequently its incident and emergent parts produced, will be nearly parallel and coincident [e], and so the points P, ϖ will nearly coincide: and when the points O, A, C coincide at a reflecting surface, the incident and reflected rays produced will make equal angles with the perpendicular $QC\varkappa$, and so the triangles PCQ, $\varpi C\varkappa$ will be equal. Thirdly, when the

Fig. 219, 220.

e Art. 42.

Fig. 217.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the eye is at the center of a reflecting concave. For then the incident and reflected rays, and consequently $\varpi\varkappa$ and $P\mathfrak{Q}$, will coincide. Fourthly, when a ray PO coming directly to the eye, makes an angle $PO\mathfrak{Q}$ equal to AOC or $\varpi O\varkappa$. For then the triangles $PO\mathfrak{Q}$, $\varpi O\varkappa$ are equal. This happens in a reflecting concave when the object is very near its center. For producing the object $P\mathfrak{Q}$ till it cuts the reflected ray in p, since the angles $PO\mathfrak{Q}$, $\varpi O\varkappa$ or $pO\mathfrak{Q}$ are supposed to be equal, the lines $P\mathfrak{Q}$, $p\mathfrak{Q}$ will also be equal, and consequently a line $\mathfrak{Q}A$ will nearly bisect the angle PAp, when A is very near to C [a]; as a line drawn from the center E will do [b], and so the points $\mathfrak{Q}$, E are almost coincident. Fig. 203, 207. 218.

a Art. 59.
b Art. 8.

143. The apparent distance of an object seen in a telescope or a microscope, is to its apparent distance perceived by the naked eye, as its apparent magnitude to the naked eye, to its apparent magnitude in the telescope or microscope. For conceiving AC to be the object-glass, and the eye to be close to the eye-glass at O; the visual ray AO will go in a manner straight through it [c]; and so the apparent magnitude and apparent distance of the object will continue the same as when there was no eye-glass: and since, when the vision is distinct, the rays in every pencil come parallel through the eye-glass, the apparent magnitude and consequently the apparent distance will still continue the same as before while the eye is drawn back [d].

Apparent and true distance of the object compared in telescopes and microscopes. Fig. 221, 222, 223.

c Art. 42.
d Art. 107.

Consequently the apparent distance in a telescope is to the apparent distance perceived by the naked eye, as the focal distance of the eye-glass, to the focal distance of the object-glass; by the 120th and 141st articles. Which may thus be demonstrated independently of the 120th article. Let pq be the image of a remote object terminated by the line PCp; so that qC and qO may be the focal distances of the object-glass and eye-glass; then supposing the object viewed by the naked eye at C, since the angles $\varpi O\varkappa$, $PC\mathfrak{Q}$ have equal subtenses $\varpi\varkappa$ and $P\mathfrak{Q}$, the apparent distance, $O\varkappa$, in the telescope, is to the apparent distance, $C\mathfrak{Q}$, to the naked eye at C, as the latter angle $PC\mathfrak{Q}$ to the former $\varpi O\varkappa$ [e], or as the opposite angle pCq, to the opposite angle pOq, or since pq subtends them both, as the latter focal distance qO, to the former qC [e].

e Art. 60.

The same proportion may be proved when AC represents the eye-glass of a telescope or microscope, and the object-glass is placed at o, the conjugate focus to O. For let pq be the image of the remote object $P\mathfrak{Q}$ terminated by the line PoA, and when qo and qC are the focal distances of the glasses at o and C, the ray AO will be parallel to pC [f]. Now the apparent distance $O\varkappa$, is to the apparent distance $o\mathfrak{Q}$, perceived by the naked eye at o, as the latter angle $Po\mathfrak{Q}$ to the former $\varpi O\varkappa$, or as the opposite angle poq to the opposite angle AOC or its equal pCq, or as the latter distance qC to the former qo.

Fig. 224, 225, 226.

f Art. 50.

144. An

Apparent diſtances of the object and glaſs compared. Fig. 197 to 226.

144. An object ſeen in any glaſs will appear behind it, or at it, or before it, according as, $\varpi\kappa$ or PQ, the real magnitude of the object is bigger, equal or leſs than AC the part of the glaſs in which it appears. For ſince $\varpi\kappa$ and AC do both ſubtend the ſame or equal angles at the eye, $O\varpi$ will be bigger, equal or leſs than OC, according as $\varpi\kappa$ or PQ is bigger, equal or leſs than AC.

Hence it follows that an object always appears behind any ſurface or glaſs which cannot make rays go parallel that diverged from the eye. For then PQ or $\varpi\kappa$ will always be greater than AC.

The rule is true in a globe or in any number of ſurfaces, taking A for the concourſe of the incident and emergent parts of the viſual ray produced, and a perpendicular AC upon the axis, inſtead of the aperture of a ſingle glaſs.

The diſtances of the real and apparent objects from the glaſs compared. Fig. 209, 210.

145. It appears by the conſtant ſimilitude in the ſhapes of the triangles $A\varpi P$, AOo, that the ratio of $A\varpi$ to AP, that is of the diſtances of the apparent and real objects from the glaſs, is the ſame as the ratio of AO to Ao, that is of the diſtances of the eye and its conjugate focus from the glaſs. Conſequently in refractions at a plane ſurface this ratio is the ſame as of the ſine of incidence to the ſine of refraction [a] of a ray coming from the object to the eye; which in refractions out of water into air is as 3 to 4; and in reflections at a plane it is a ratio of equality [b].

a Art. 31.

b Art. 23.

Sometimes an object appears in the place of its image.

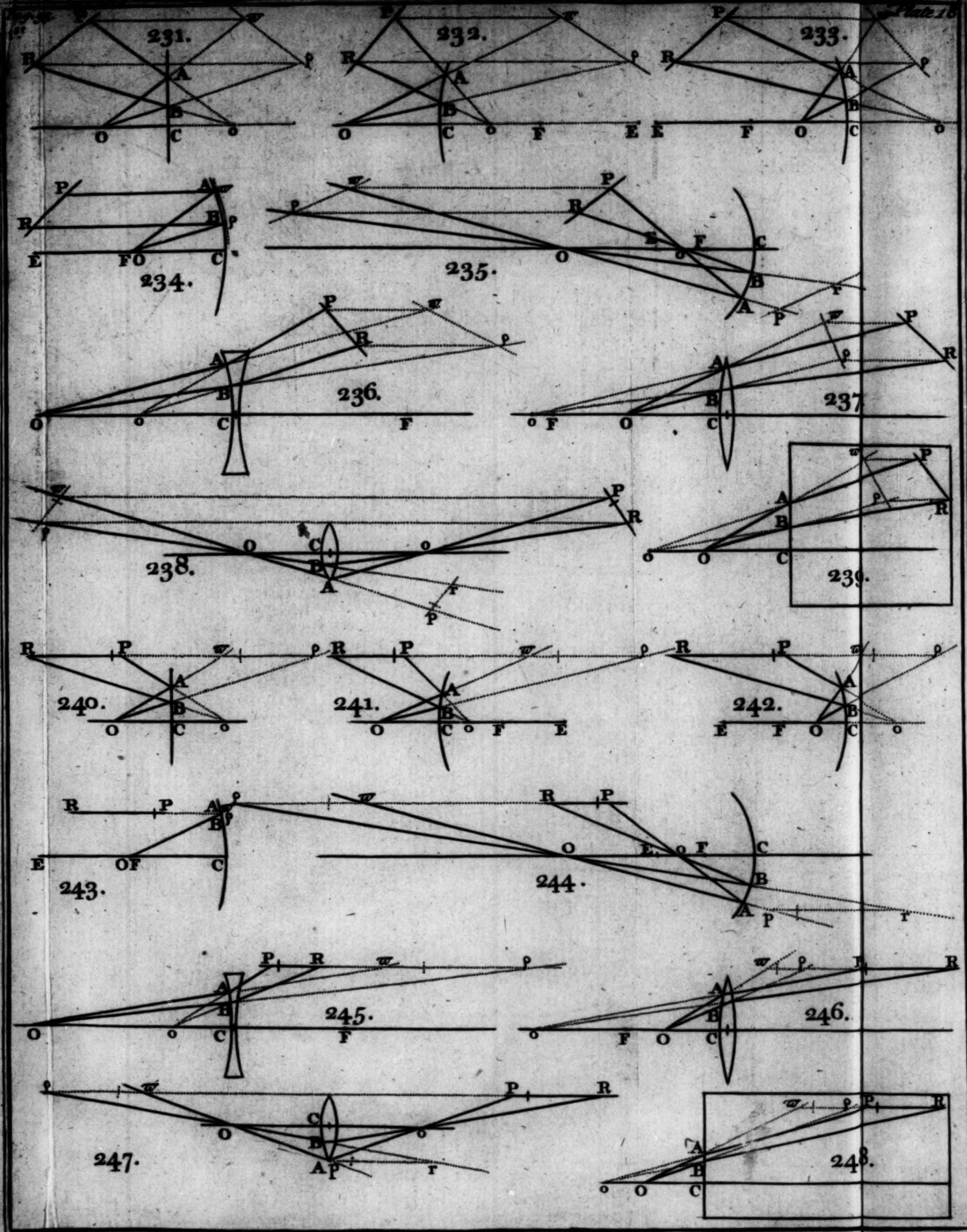
146. Therefore in theſe two caſes the object appears in the place of its image; not becauſe the rays diverge from that place to the eye, which the ſenſe does not perceive, but becauſe the object is equal to the image; and conſequently its apparent magnitude and diſtance are the ſame as if it was put in the place of the image and viewed there by the naked eye. But if an object be put in the place of an image which is leſs than it ſelf, it will appear bigger [c] and conſequently nearer to the naked eye than it did in the glaſs [d]; that is, the object in the glaſs will appear remoter than the place of its image: and on the contrary. And in general, the apparent diſtance of an object, is to the real diſtance of its laſt image, as the real magnitude of the object, to the real magnitude of that image. Becauſe the apparent object ($\varpi\kappa$) and the laſt image, ſubtend the ſame angle at the eye.

c Art. 105.

d Art. 138.

How the apparent diſtance varies while the glaſs and eye are fixt and object is moved. Fig. 197 to 220.

147. While the glaſs and eye are fixed, and the object is gradually removed from the glaſs, we may ſuppoſe the lines OA, Ao to be fixt and only the parallel $P\varpi$ to be moveable; and from hence it will be evident, eſpecially from the conſtant ſimilitude in the ſhape of the variable triangle $PA\varpi$, to the fixt triangle oAO, that, in all glaſſes which cannot make the ray AP go parallel to the axis, while AP increaſes, $A\varpi$ and $O\varpi$ will alſo increaſe perpetually, wherever the eye is fixt; and that $O\varpi$ will alſo increaſe perpetually in any other glaſs that can make the ray AP go parallel to the axis, when the eye is fixt between this glaſs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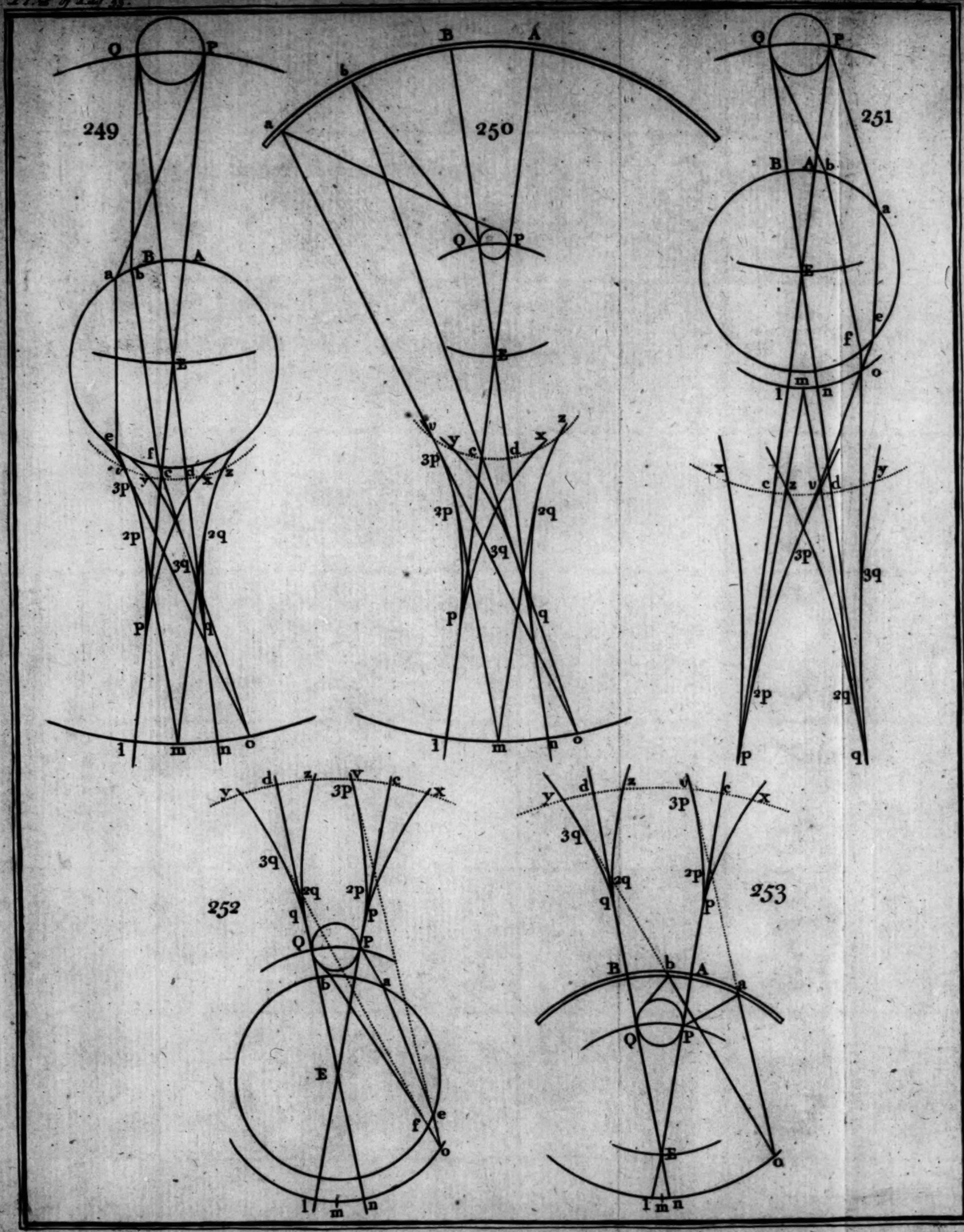
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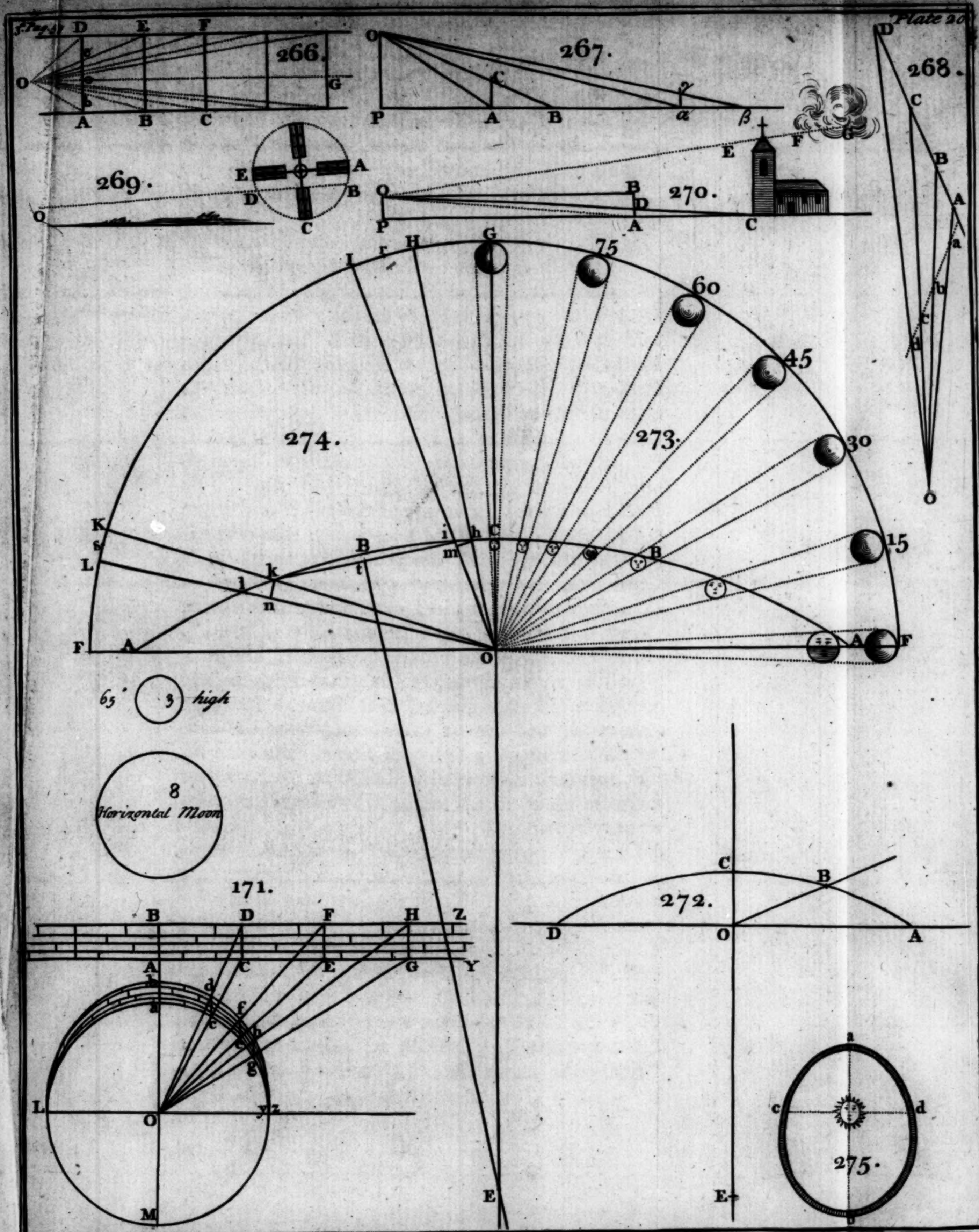


Plate 20
266.
267.
268.
269.
270.
274.
273.
75
60
45
30
15
65° ☽ high
8
Horizontal Moon
171.
272.
275.

glaſs and its principal focus. But if the eye be fixt at this focus, $A\varpi$ being nothing, $O\varpi$ will be conſtantly equal to its focal diſtance; and when the eye is fixt on this ſide the focus, $O\varpi$ will decreaſe till P arrives at o, and after it has paſſed over o, $O\varpi$ will increaſe perpetually till it equals OA, when oP equals oA; and alſo till it equals OP, when the angle POQ becomes equal to $\varpi O\kappa$ or AOC; that is when the true and apparent magnitudes of the object become equal.

How it varies when the glaſs and object are fixt and the eye is moved.

148. There will be the like variations of apparent diſtance while the glaſs and object are fixt and the eye is gradually removed from the glaſs; that is in all ſorts of glaſſes and ſurfaces which cannot make diverging rays become parallel, while AO increaſes, $O\varpi$ will alſo increaſe perpetually wherever the object be fixt; and in any other glaſs that can make them parallel, $O\varpi$ will alſo increaſe when the object is fixt between this glaſs and its principal focus. But if the object be fixt at this focus, $O\varpi$ will be conſtantly equal to its focal diſtance; and when the object is fixt on this ſide the focus, $O\varpi$ will decreaſe till o arrives at Q, but after it has paſſed over Q, $O\varpi$ will increaſe perpetually; till it becomes equal to OA and then to OP as in the former article.

For the glaſs and object being fixt, the image of the object is alſo fixt in place and magnitude; and being beyond the glaſs in the two firſt caſes, the angle it ſubtends at the eye will continually decreaſe, while the eye recedes from it and from the glaſs; and conſequently the given object $\varpi\kappa$ will ſubtend that decreaſing angle at greater and greater diſtances from the eye: but when the object is at the principal focus, the angle which meaſures its apparent magnitude will be invariable; and conſequently $O\varpi$ will alſo be invariable and equal to the focal diſtance; and when the object is farther from the glaſs than its focal diſtance, its image will be on this ſide of the glaſs, and the eye in receding from the glaſs will firſt come nearer to the fixt image till it paſſes by it, and then recedes from it; ſo that the apparent magnitude will firſt increaſe and then decreaſe, and conſequently the apparent diſtance will firſt decreaſe to nothing, and then increaſe perpetually[a].

See Plate 13.

Two perſons viewing one another through a lens apppear at the ſame diſtance from each other.

149. Two perſons NO, PQ viewing one another through any given lens AC, appear at equal diſtances from one another. For let two rays PAO, NAQ croſs one another at any point of the lens and let the viſual rays OA, QA produced meet the parallels $P\varpi$, $N\nu$ in ϖ and ν, and the perpendiculars $\varpi\kappa$, $\nu\omega$ will be the apparent objects[b]. Now ſince the bendings of the rays NAQ, PAO are equal[c], the angles NAO, PAQ are alſo equal, and being but ſmall, NO is to PQ (as AO to AQ[d] or) as the angle AQC to the angle AOC[e]: that is the apparent objects $\nu\omega$, $\varpi\kappa$ are proportionable to the angles they ſubtend at the eyes Q, O, and conſequently their diſtances from them are equal, as in all caſes of viſion with the naked eye.

Fig. 227. to 230.

a Art. 140.
b Art. 139.
c Art. 45.
d Art. 57.
e Art. 60.

150. Hence

At any two places of the lens equidistant from the persons.

150. Hence when the glass is at any two places C, D equidistant from the extremities of the given interval OQ, the object will appear at the same distance to the same eye. For $O\varkappa$, the apparent distance of the object PQ, being equal to, $Q\nu$, the apparent distance of the object NO seen through the glass at the distance QC, will also be equal to the apparent distance of the object PQ, seen through the glass removed to the distance OD equal to QC.

How that apparent distance varies while the eye and object are fixt and the glass is moved.

151. When the interval between the eye and object is fixt and a concave lens is gradually moved from either end of it to the other, the apparent distance of the object will first increase and then decrease again; and will be the greatest of all when the glass is exactly in the middle of that interval. But when a convex lens is carried from either end to the other, the apparent distance of the object will first decrease and then increase and will be the least of all when the lens is exactly in the middle of the interval, provided it be less than 4 times the focal distance of the lens; but if it be equal to 4 focal distances, the apparent object will seem to touch the eye, being infinitely great and infinitely confused, when the glass is in the middle. And when the interval between the eye and object is bigger than 4 focal distances, the object will appear infinitely great and confused and consequently infinitely near to the eye, when the lens is at two places, suppose at C and D equidistant from the eye and object; so that while the lens is carried from either end to the other the apparent distance will first decrease, and then increase, till the lens gets to the middle, and then will decrease and increase again till the lens gets to the end: and when the lens is in the middle the apparent distance of the object will be less, equal or greater than its true distance, according as the whole interval is less, equal or greater than 8 focal distances of the lens; and consequently if greater than 8, the apparent distance will be equal to the true distance, when the lens is at two places between C and D equidistant from them and from the middle: and all this while the apparent magnitude of the object will increase when its apparent distance decreases, and on the contrary[a]; as any one will find that pleases to make the experiment. And the reason of all these appearances is shewn by an easy rule in the next book.

a Art. 140.

Appearances of inclined objects determined. Fig. 231 to 239.

152. When an object PR is inclined to the axis of any glass, its apparent inclination may be determined, as before[b], by drawing the lines $P\varpi$, $R\varrho$ parallel to the axis or unrefracted ray OC, till they meet the rays, OA, OB, by which the points P and R are seen, in ϖ and ϱ; and by drawing the line $\varpi\varrho$, which will be the apparent object. Because its extremities ϖ, ϱ, viewed by the naked eye, are the apparent places, in the glass, of the extremities of two other objects conceived to touch the extremities of the inclined object PR, and to stand perpendicular to the axis of the glass[c]: observing, as before, that when PR and AB are on con-

b Art. 139.

c Art 139.

contrary sides of the axis, the naked eye must either be inverted and turned about to view the inclination of $\varpi\varrho$, or else two other distances Op and Or must be taken equal to $O\varpi$ and $O\varrho$ in opposite parts of the same rays produced; and then if the glass be removed, the line pr will appear in the same place and position as the object PR appeared in, in the glass.

153. Hence if an object PR be parallel to the axis of the glass, produce it till it cuts the visual rays OA, OB in ϖ and ϱ; and the line $\varpi\varrho$ will appear to the naked eye in the same place and position as PR appeared in, in the glass. It is to be observed that though the real places of the lines PR, $\varpi\varrho$ are parallel to the axis, yet they do not appear so to the naked eye; but seem to converge towards the remoter parts of the axis, for a reason to be mentioned in Art. 156.

And of objects parallel to the axis. Fig. 240 to 248.

By the description of causticks in Art. 69 &c. and by the figures there explained, it is easy to understand, that the edge of a thin plate may be formed into a curve of such a shape and degree of convexity, that when it is applyed within the concavity of a leg of a given caustick, it shall touch every ray in a different point of its convexity *. This convex edge it self may be called the leg of the caustick; and is represented in the following figures by the curve $p\,3\,pv$ or $q\,3\,qy$, &c.

Definition. Fig. 131. &c.

* See Art. 445, 446.

154. Hence when the eye is fixt at any point o, placed any where but in the very curve of a given caustick, formed by refractions or reflections of all the rays that flowed from P; the visual ray * by which the point P is then seen, may be found by drawing a line from the eye at o, that shall touch a leg of the caustick in a single point $3\,p$, without cutting the same leg. And if you conceive a fine line or string to be fixt at the farthest end of this leg from the eye, and to be lapped upon a part of its convexity, and thence to be extended from it in a straight line $3\,po$, the point P will always appear in the successive directions of this string, while the eye, or the caustick it self, is moved sideways.

Visual rays are tangents to the leg of a caustick. Fig. 249 to 253.

* Art. 90.

155. If a small round object be seen inverted through a sphere, or in so large a portion of a reflecting concave surface as to form a caustick, it will appear biggest and nearest to the eye placed in a line drawn through the object and the center of the sphere or spherical surface: and the object will appear gradually smaller and remoter while the eye, or the sphere, or the object it self, is moved sideways: and the contrary appearances may happen in a lesser degree when the object is seen upright, in the cases specified by the figures, which among other ways may be thus demonstrated.

How the apparent magnitude of an object varies by moving the eye, object or glass sideways.

Through the center E of the sphere or speculum, draw two lines EP, EQ touching the opposite sides of the small round object PQ; and of all the rays that flow from P, let the nearest to the line PEp, belong to the focus p after refractions or reflections; and the rest of them will be formed into a caustick, whose legs pv, px are always convex towards PEp, the axis of the pencil *. Also let qy and qz be the legs of another caustick formed by the

Fig. 249 to 253.

* Art. 69. &c.

pencil flowing from Q, whose axis is QEq. Then with the center E, and any semidiameter El, draw an arch $lmno$, cutting the (produced) axes Pp, Qq in l and n; and from the eye first placed at m within the angle lEn under the axes, draw the lines $m2p$, $m2q$ so as to touch a leg of each
* Art. 154. caustick; and the object PQ will appear under the visual angle $2pm2q$*; Again from the eye removed to any point o placed out of the angle lEn under the axes, draw two other lines $o3p$, $o3q$ touching a leg of each caustick at $3p$ and $3q$; and the object PQ will now appear under the
* Art. 154. visual angle $3po3q$*,

Now while the eye is moving sideways in the arch mno, one of the points of contact $2p$ will move continually in the same leg from $2p$ to $3p$; but the other point $2q$ will move first from $2q$ to q in the same leg, and then will return along the other leg of the same caustick from its cusp q to the point $3q$. And so the visual rays $3po$, $3qo$ will now come to the eye at o, from those legs of the two causticks that lye both on the same sides of their axes Ep, Eq.

Let any circle described upon the center E, cut the two last mentioned legs $3pp$, $3qq$, in the points v and y, and their respective axes Ep, Eq in c and d; and since the two causticks, upon account of the equal distances EP, EQ, resulting from the roundness of the object PQ, are equal to each other, it is easy to understand that the arch cv is equal to dy, and consequently that the arch vy between these two legs, will be equal to the corresponding arch cd between their axes. And the like property being true in every circle described about the same center E, it appears that these legs approach towards each other in approaching towards the center E.

Fig. 249, 250. Therefore when the object appears inverted to the eye any where in
* Art. 103. the arch mno*, it will appear biggest when the eye is between the axes Ep,
* Art. 138. Eq, and gradually smaller, and consequently remoter*, while the eye is moving sideways, because the visual angle decreases. And it is not difficult to understand by considering the 251st, 252d, and 253d figures, that the contrary may happen when the object appears upright.

It is easy to apprehend that there will be a like variety of appearances, when the eye stands still and the object is moved sideways in a circle PQ whose center is E; and also when the center E of the sphere or speculum, is moved sideways in a circle whose center is at the object,

Parallel lines seen obliquely appear to converge. Fig. 266.

156. Parallel lines seen obliquely, as ABC, DEF, appear to converge more and more as they are farther extended from the eye. Because the apparent magnitudes of their perpendicular intervals AD, BE, CF, &c. are perpetually diminished. And for the same reason they appear to converge towards an imaginary line OG drawn from the eye parallel to them.

This is the reason that the remoter parts of a walk or a floor appear to ascend gradually, and the cieling to descend towards the horizontal

line

line OG: and that the ſurface of the ſea, ſeen from an eminence, appears to aſcend gradually in going from the ſhore; and that the upper parts of very high buildings ſeem to lean forward over the eye below; becauſe they ſeem to approach towards a vertical line OG.

157. The apparent magnitude of a given line, AB, ſeen very obliquely at a given diſtance, OA, increaſes and decreaſes in proportion to the increaſe and decreaſe of, OP, the perpendicular diſtance of the eye from the line AB produced; provided the diſtance AO be very large in compariſon to AB. For let the ray BO cut a line AC perpendicular to AB in C; and while the eye is raiſed or depreſſed in the perpendicular OP, the line AC will increaſe and decreaſe as OP does, and ſo will the angle AOC ſubtended by AC[a], and this angle meaſures the apparent magnitude of AB[b].

How the apparent magnitude of an oblique object varies. Fig. 267.

a Art. 59.
b Art. 98.

Hence the apparent magnitudes of equal parts AB, $\alpha\beta$ of a line $PA\sigma$, ſeen very obliquely at great diſtances from the eye, are reciprocally in a duplicate proportion of thoſe diſtances. For example let $O\beta$ be double of OB, and the angle OBP will be double of $O\beta P$[c], and accordingly ſince AB, $\alpha\beta$ are equal, the perpendicular AC will be double of $\alpha\gamma$ and being ſeen twice as near as $\alpha\gamma$, will appear four times bigger than $\alpha\gamma$. Again if $O\beta$ be treble of OB, the line AC will be treble of $\alpha\gamma$, and being ſeen three times nearer than $\alpha\gamma$, will appear nine times bigger than $\alpha\gamma$; and ſo on.

c Art. 60.

Hence the apparent intervals between a row of columns are diminiſhed in a greater proportion than their apparent heights.

158. This quick diminution of the apparent magnitudes of the remoter parts of long lines or diſtances, is the cauſe of great difficulty and uncertainty in our eſtimate of their quantities. For be the differences of ſeveral diſtances or heights never ſo great in themſelves, they will become inviſible at laſt by reaſon of the ſmalneſs of the angles they ſubtend at the eye, occaſioned by their obliquity: and then thoſe unequal heights and diſtances will appear equal.

Why unequal diſtances appear equal.

159. Diſtances from the eye ſeen upon a rough, uneven ſurface appear ſhorter, than if it was perfectly plane. For the inequalities of the ſurface, ſuch as hills and holes and rivers that lye low and out of ſight, either do not appear or hinder the parts from appearing that lye behind them; and ſo the whole apparent diſtance is diminiſhed by the parts that do not appear in it. It is a common obſervation that the banks of a river appear contiguous to a diſtant eye when the river is low and is not ſeen: inſomuch that travellers in a ſtrange place are frequently uncertain where the river runs, and whether the objects they ſee before them are on this ſide or on that ſide of it. And when a flag or weather-cock appears above any high building, the ſight alone at a moderate diſtance cannot diſtinguiſh whether it belongs to that building or to ſome other

And ſometimes ſhorter ſometimes longer.

 behind

behind it. In like manner the ſun, moon, and clouds, and the tops of mountains, and all objects in the horizon, when ſeen in the ſame direction appear all at the ſame diſtance.

Fallacies in viſion.

160. The four laſt articles afford a ſolution of ſeveral fallacies in viſion, ſome of which I have here collected. Since oblique diſtances appear longer in proportion as the eye is raiſed higher, to view them more fully; it follows that being placed at a diſtance from a gentle aſcent, like the ſtage at a play-houſe, or a riſing mount at the end of a walk, we ſhall judge thoſe aſcents much longer than if they were level; eſpecially if they be artfully contracted in the remoter parts. For by not obſerving, or attending to, the reality of theſe aſcents, we form the ſame idea of them, as is uſually ſuggeſted to the mind by a longer level walk with parallel ſides. Now ſince the riſing of the ground together with a gradual diminution of its breadth, when not obſerved, do make it ſeem longer and conſequently leſs diminiſhed in breadth than if the ſame extent was level, with parallel ſides; it ſhould follow that a gentle aſcent alone, whoſe ſides are parallel, ſhould ſtill widen the appearance of their remoter parts, ſo as to make the parallel ſides appear parallel or even diverging; which is contrary to the common appearance of parallel ſides. A deception of this kind may be ſeen in a Viſta of parallel rows of trees, when viewed at the front of the Honourable Mr. *North*'s houſe at *Rougham* in *Norfolk*; as I am informed by a neighbour of his, my worthy friend *Martin Folkes* Eſq; whoſe great knowledge and curioſity lets nothing eſcape him. Being aſſured by Mr. *North* that the trees were parallel, which ſeemed to diverge, he was much ſurprized at this uncommon appearance; till after a little conſideration, he perceived that the cauſe of the deception was a gentle riſing of the ground where the trees were planted, and a gentle deſcent for half a mile from the houſe to the beginning of the plantation.

He has alſo told me, that upon coming into a ſtreet, in a dark night, where there was but one row of lamps, he has often miſtaken the ſide of the ſtreet they were in; which he accounts for in this manner. Let *O* be the ſpectator, *A*, *B*, *C*, *D*, the lamps on his right hand; *AaO*, *BbO*, *CcO*, *DdO* the rays that come to his eye. Now if he happens to imagine the neareſt lamp *A* to be the remoteſt, ſuppoſe at *a*; he will conſequently imagine all the reſt to be at *a*, *b*, *c*, *d* in a contrary ſituation of a line extended on his left hand.

Fig. 268.

The oblique ſituation of an object ſeen alone is ſuggeſted to the mind by a greater apparent magnitude or a diſtincter perception of the nearer than of the remoter parts. And conſequently if the object be ſo remote or ſo uniform that we are not affected with a ſenſible difference in thoſe perceptions, we are ſubject to miſtake its poſition. For an object may appear under the ſame angle *AOD* in two oblique poſitions *AD* and *ad*.

Fig. 268.

Hence

Hence we ſometimes miſtake the poſition of a weather-cock or a flag; and by taking the neareſt end of the ſail of a wind-mill for the remoteſt, we ſometimes miſtake the courſe of its circular motion. For if a ſpectator at *O*, ſituated nearly in the plane of the ſails produced, imagines the fartheſt end *A* of a ſail *AE* to be the neareſt, and the real motion of the ſails be in the order of the letters *ABCDE*; when *A* is moved to *B* and the line *BO* is drawn, cutting the circle *ABCDE* in *D*; ſince he firſt imagined the end *A* to be at *E*, he will not now conceive it at *B* but at *D*; and ſo will imagine the courſe of the motion to be from *E* to *D*; which is contrary to the real motion from *A* to *B*. The uncertainty we ſometimes find in the courſe of the motion of a branch or hoop of lighted candles, turned round at a diſtance, is owing to the ſame cauſe: and alſo that we miſtake a convex for a concave ſurface ſometimes with the naked eye, but more frequently in viewing ſeals and impreſſions with a convex glaſs or a double microſcope; and hills and valleys in the moon with teleſcopes, eſpecially if they invert the object: being led into the miſtake by an imperfect judgment of the diſtances of the parts of the object, and confirmed in it by a contrary poſition of the ſhadows caſt by a ſide light. Fig. 269.

We are frequently deceived in our eſtimates of diſtance by any extraordinary magnitudes of objects ſeen at the end of it: as in travelling towards a large city or a caſtle or a cathedral church or a mountain larger than ordinary, we think they are much nearer than we find them to be upon trial. For ſince by experience the ideas of certain quantities of known diſtances are uſually annexed to the apparent magnitudes of known objects of a common ſize; and ſince the apparent magnitudes of thoſe larger objects at a greater diſtance are the ſame as of the ſmaller at a ſmaller diſtance, it is no wonder they ſuggeſt the uſual idea of ſmaller diſtance annext to more common objects. This is farther evident, becauſe we are ignorant of the nature of the country; and of the inequalities in the ground interpoſed.

Animals and all ſmall objects ſeen in valleys, contiguous to large mountains, appear extraordinary ſmall; becauſe we think the mountain is nearer to us than if it was ſmaller; and we ſhould not be ſurprized at the ſmalneſs of the neighbouring animals if we thought them farther off. In like manner when they are placed upon the top of the mountain or upon a large building and are viewed from below, we think they are extraordinary ſmall for the ſame reaſon, and alſo becauſe we judge the mountain or the building to be lower in proportion than if it was ſmaller; both becauſe of its extraordinary magnitude and greater obliquity of its higher parts to the viſual rays. *Dechales* tells us that while he ſtood at the bottom of a mountain, he once obſerved a parcel of crows going to fly over it, which at firſt he thought were higher than the moun-

mountain; because, I suppose, they appeared so very small in comparison to it; but he found they spent half an hour in ascending before they got to the top of it [1]. The part of the *Monument* extant above the tops of the adjoining houses, I am told, is 5 times longer than the height of the houses, and yet from below that part appears but two or three times longer at most; because of its unusual magnitude and obliquity to the sight.

Aguilonius mentions a fallacy in distance which he had frequently observed and admired. In a warm summers morning when fogs are exhaled from moist ground, we frequently see them very near us in some known place; but so soon as they are separated from the ground and are going to ascend they appear so remote, that, says he, I could never have believed they hung over that place, had I not seen them there but the moment before [2]. The reason is they then appear in the manner and direction of other remote clouds in the horizon; whose difference in distance cannot be discerned, for want of some visible surface extended between them; like the surface of the ground when the rising cloud lay upon it.

It is said to be a common observation made by travellers in the night or the dusk of the evening, that near objects, as trees and houses are often taken to be very large and remote. The reason may be, that being unable to discern the quantity of ground interposed, they refer them to the brighter sky in the horizon, and so think they are remoter and consequently larger: as I remember a red coat of arms, upon the top of an iron gate at the end of a walk, was taken for a brick house in the fields beyond it.

The greatest quantity of apparent distance determined. Fig. 270.

161. If the surface of the earth was perfectly plane, the distance of the visible horizon from the eye would scarce exceed 5000 times the height of the eye above the ground, or the distance of 5 miles supposing the height of the eye between 5 and six foot: and all objects placed beyond this distance would appear in the visible horizon. For let *OP* be the height of the eye above the line *PA* drawn upon the ground; and if an object *AB*, equal in height to *PO*, be removed to a distance *PA* equal to 5000 times that height, it will hardly be visible by reason of the smalness of the angle *AOB* [a]. Consequently any distance *AC*, how great soever, beyond *A*, will be invisible. For since *AC* and *BO* are parallel, the ray *CO* will always cut *AB* in some point *D* between *A* and *B*; and therefore the angle *AOC* or *AOD* will always be less than *AOB*, and therefore *AD* or *AC* will be invisible. Consequently all objects and clouds, as *CE* and *FG*, placed at all distances beyond *A*, if

a Art. 97.

1 *Dechales* Cursus Math. Tom. 3. p. 435.
2 *Aguilonii* Optica p. 223.

they

they be high enough to be viſible or to ſubtend a bigger angle at the eye than *AOB*, will appear at the horizon *AB*; becauſe the diſtance *AC* is inviſible.

162. Hence if we ſuppoſe a vaſt long row of objects, or a vaſt, long wall, *ABZY*, built upon this vaſt plane, and its perpendicular diſtance, *OA* from the eye at *O*, to be equal to, or greater than, the diſtance *Oa* of the viſible horizon, it will not appear ſtraight but circular, as if it was built upon the circumference of the horizon *acegy*: and if the wall be continued to an immenſe diſtance, its extream parts *YZ* will appear in the horizon at *yz* where it is cut by a line *Oy* parallel to the wall. For ſuppoſing a ray *YO*, the angle *YOy* will become inſenſibly ſmall. Imagine this infinite plane *OAYy*, with the wall upon it, to be turned about the horizontal line *Oy*, like the lid of a box, till it becomes perpendicular to the other half of the horizontal plane *LMy*, and the wall parallel to it, like a vaſt ceiling over head; and then the wall will appear like the concave figure of the clouds over head. But though the wall in the horizon appeared in the ſhape of a ſemicircle, yet the ceiling will not, but much flatter. Becauſe the horizontal plane was a viſible ſurface, which ſuggeſted the idea of the ſame diſtances quite round the eye, but in the vertical plane extended between the eye and the ceiling, there is nothing that affects the ſenſe with an idea of its parts but the common line *Oy*, conſequently the apparent diſtances of the higher parts of the ceiling will be gradually diminiſhed in aſcending from that line. Now when the ſky is quite overcaſt with clouds of equal gravities, they will all float in the air at equal heights above the earth, and conſequently will compoſe a ſurface, reſembling a large ceiling, as flat as the viſible ſurface of the earth. Its concavity therefore is not real but apparent: and when the heights of the clouds are unequal, ſince their real ſhapes and magnitudes are all unknown, the eye can ſeldom diſtinguiſh the unequal diſtances of thoſe clouds that appear in the ſame directions, unleſs when they are very near us, or are driven by contrary currents of the air. So that the viſible ſhape of the whole ſurface remains alike in both caſes. And when the ſky is either partly overcaſt or perfectly free from clouds, it is matter of fact we retain much the ſame idea of its concavity as when it was quite overcaſt. But if any one thinks that the reflection of light from the pure air, is alone ſufficient to ſuggeſt that idea, I will not diſpute it.

Apparent concavity of the ſky explained. Fig. 271.

163. The concavity of the heavens appears to the eye, which is the only judge of an apparent figure, to be a leſs portion of a ſpherical ſurface than a hemiſphere: I mean that the center of the concavity is much below the eye, and by taking a medium among ſeveral obſervations, I find the apparent diſtance of its parts at the horizon is generally between three and four times greater than the apparent diſtance of its parts

Apparent concavity of the ſky determined.

Fig. 272. parts overhead. For let the arch *ABCD* represent the apparent concavity of the sky, *O* the place of the eye, *OA* and *OC* the horizontal and vertical apparent distances, whose proportion is required. First observe when the sun or the moon or any cloud or star is in such a position at *B*, that the apparent arches *BA*, *BC*, extended on each side of this object towards the horizon and zenith, seem equal to the eye; then taking the altitude of the object *B* with a quadrant, or a cross-staff, or finding it by astronomy from the given time of observation, the angle *AOB* is known. Drawing therefore the line *OB* in the position thus determined, and taking in it any point *B* at pleasure, in the vertical line *CO* produced downwards seek the center *E* of a circle *ABC*, whose arches *BA*, *BC*, intercepted between *B* and the legs of the right angle *AOC*, shall be equal to each other; then will this arch *ABCD* represent the apparent figure of the sky. For by the eye we estimate the distance between any two objects in the heavens by the quantity of sky that appears to lye between them; as upon earth we estimate it by the quantity of ground that lyes between them. The center *E* may be found geometrically by constructing a cubick equation, or as quick and sufficiently exact by trying whether the chords *BA*, *BC*, of the arch *ABC* drawn by conjecture, are equal; and by altering its radius *BE* till they are so. Now in making several observations upon the sun and some others upon the moon and stars, they seemed to me to bisect the vertical arch *ABC* at *B*, when their apparent altitudes or the angle *AOB* was about 23 degrees; which gives the proportion of *OC* to *OA* as 3 to 10 or as 1 to $3\frac{1}{3}$ nearly. When the sun was but 30 degrees high, the upper arch seemed always less than the under, and I think always greater when the sun was about 18 or 20 degrees high.

Why the sun and moon appear bigger near the horizon than higher up. Fig. 273.

164. I have been the more particular in considering the apparent figure of the sky, because I do not find it has ever yet been determined; although it be absolutely necessary to a satisfactory solution of several noted appearances in the heavens. For instance, supposing the arch *ABC* to represent that apparent concavity, I find the diameter of the sun or moon will seem to be greater in the horizon than at any proposed altitude, measured by the angle *AOB*, in the proportion of its apparent distances *OA*, *OB*. The numbers that express these proportions are set down in this table, over against the corresponding altitudes of the sun or moon, and are also exactly represented to the eye in the 273d figure, in which the moons placed in the quadrantal arch *FG*, described about the center *O*, are all equal to each other, and represent the body of the moon at the heights there noted;

The sun or moon's altitudes in degrees.	Apparent diameters or distances.
00	100
15	68
30	50
45	40
60	34
75	31
90	30

and

and the unequal moons in the concavity *ABC* are terminated by the visual rays that come from the circumference of the real moon, at those heights, to the eye at *O*. The diameters of these unequal moons at *A* and *B* do therefore bear the same proportion to each other as their apparent distances *OA*, *OB*[a]; and they must appear in the very same proportion that they really have in this concave, because we judge all objects in the heavens to be in this very surface[b]: and so the appearance to the eye is exactly the same as if several moons were painted upon a real surface *ABC* in the proportions here assigned; in which case we should certainly judge the real magnitudes of the larger paintings of the lower moons to be really larger; though the visible magnitudes of them all, answering to their equal images upon the retina, were exactly equal. And let any one consider when he looks at the moon, whether it does not seem to be a real, tangible substance, as well as all other objects we look at.

a Art. 57.

b Art. 161, 162.

This solution confirmed by observations of stars.

165. For the same reason all other objects, and distances of stars in the heavens, as well as the sun and moon, must seem to be greater in the horizon than in higher situations; and it is well known they do so. Hence I deduce another experimental proof, that these proportions of the moon's apparent magnitudes are exactly assigned. In a clear star-light night take notice of the distance of any two stars that lye very near each other, and as high as possible; and at the same time pick out two other stars situated as low as possible, whose distance from each other seems equal to that of the two higher stars. Then by a globe, map, or by calculation, find the real distances of each pair of stars in degrees and minutes and also the altitudes of the middle points of those distances above the horizon; and take the arches *Fr*, *Fs* equal to them; and set off the arches *rH* and *rI* each equal to half the distance of the higher stars, and likewise *sK* and *sL* each equal to half the distance of the lower. Then from *O* to the points *H*, *I*, *K*, *L* draw lines cutting the concave *ABC*, already determined by the method of bisection[c], in the points *h*, *i*, *k*, *l*. These points, if each couple of stars were in a vertical circle, would be their apparent places; and if their situation be not vertical, yet the perpendicular subtenses *hm*, *kn* (of the angles *hOi*, *kOl*) which in stars very near together are the measures of their apparent distances, are not altered by their oblique situation. Now by several observations and constructions I have found these subtenses *hm*, *kn* to be nearly equal to one another; and since they appeared so in the heavens, it shews that the concave *ABC* was determined right. Therefore if *HI* and *KL* were the real diameters of two unequal moons, they would appear equal at *h* and *k*, and consequently if the lower moon at *KL* be increased till it equals the higher at *HI*, then the angles *kOn*, *hOm* being now equal, the

Fig. 274.

c Art. 163.

subtense *kn* will be greater than *hm* in the ratio of their apparent distances *Ok* to *Oh* [a], which is the thing we asserted of the real moon.

a Art. 57.

Figure of the sky determined another way.

166. Hence we have another method for finding the apparent figure of the heavens by the foregoing observations of the stars. Assume one of the distances from the eye, as *Ok*, of any length you please, and take the other *Oh* in proportion to *Ok* as *KL* to *HI*, and joining *hk* bisect it in *t*, and draw a line *tE* perpendicular to it till it meets *CO*, produced, in *E*; this will be the center of the apparent concave; which I need not stay to demonstrate. In the distances *HI*, *KL*, especially in the latter, if the two stars be in a vertical circle or near it, allowance should be made for the refraction of the air.

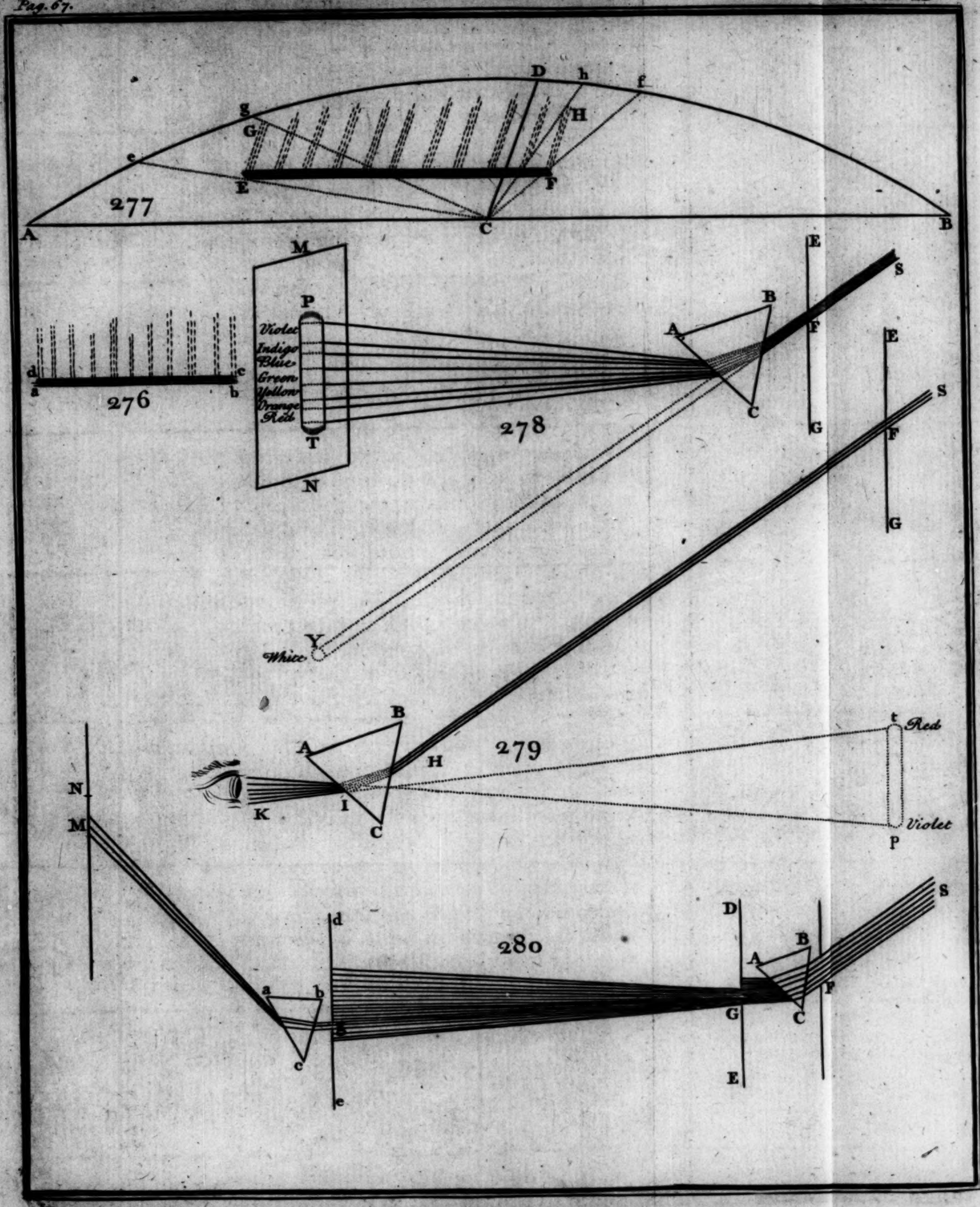
The like appearances solved in the rain-bow and halos.

167. This apparent concave, being less than a hemisphere, is also the cause that the breadths of the colours in the inward and outward rain-bows, and the interval between the bows, appear least at the top and greatest at the bottom, and in descending from top to bottom are gradually increased, though the angles subtended at the eye by all those breadths are the same in every part of the bows: and by an estimate of the apparent breadths of the inward rain-bow at two different heights, made by a friend, I determined the apparent concavity of the sky to be much the same as by the former methods. And I take it to be owing to the same cause that a halo about the sun or moon does not appear circular and concentrick to the sun or moon but oval and excentrick, with its longest diameter perpendicular to the horizon, and extended from the moon further downwards than upwards, as Sir *Isaac Newton* has described one in his opticks, page 290. For it appears by *Hugen*'s theory of halos, explained hereafter, that the rays which cause their visible appearance compose the surface of a cone, whose section made by a plane perpendicular to the ray that comes to the eye from the sun or moon, is circular and concentrick to the sun or moon; and therefore an oblique section of it made, as it were, by the apparent concavity of the sky, which is the same as the perspective projection of it upon that concave, must be such an oval figure as Sir *Isaac Newton* has described. This oval figure is also taken notice of in a halo observed by Mr. *Whiston* [1], and its excentricity has been observed by Dr. *Halley* [2]: and I have lately observed the same things my self even when the moon was very high [3]. Now since the angle which the diameter of a halo subtends at the eye is always observed to be 45 or 46 degrees, I reckon that when the bottom of the halo is near the horizon, and consequently its apparent figure is most oval, the apparent vertical diameter *ab* is divided by the moon in the proportion of about 2 to 3 or 4, and is to the horizon-

Fig. 275.

1 Phil. Transf. N°. 369. p. 214. l. 6.
2 Ibid. p. 211. l. 23.
3 21. Dec. 1729. at 7. p m.

tal


277
276
278
Violet
Indigo
Blue
Green
Yellow
Orange
Red
White
279
Red
Violet
280

tal diameter *cd*, drawn through the moon, as 4 to 3, pretty nearly. The oval in the figure is drawn according to theſe proportions to be compared with the appearance of a halo when it ſhall happen.

168. What has been ſaid of the oval projection of a halo, is applicable to the ſun or moon, whoſe projections are likewiſe oval, eſpecially near the horizon; but whether they appear ſo or not, is hard to judge, becauſe this oval projection is ſo very ſmall and ſo remote, that we cannot well perceive a ſenſible difference in the diſtances of its upper and under limbs from the eye, and conſequently can judge of its figure by no other perception but that of the figure of its picture upon the retina. On the contrary the ſun in the horizon has been often obſerved to appear oval in a contrary poſition [1], and by the preſent tables of refractions in the air, the angle ſubtended by its horizontal and vertical diameters are to each other in the ratio of 5 to 4 reſpectively, or thereabouts; becauſe the loweſt ray is more refracted than the higheſt. By this means the picture of the ſun upon the retina becomes oval, and cauſes this oval appearance. Figure of the ſun and moon conſidered.

169. The preſent theory is alſo confirmed by the appearances of the tails of comets; which, whatever be their real figure, magnitude and ſituation in abſolute ſpace, do always appear to be an arch of the concave ſky. Upon the whole it appears to me that the judgments we make of the apparent place, magnitude, ſhape, and poſition of all remote objects in the heavens, as of the ſun, moon, comets, conſtellations, rainbows, halos and all other meteors, are the very ſame as they would be if we viewed their perſpective draughts traced out by the viſual rays upon a real ſurface in the place and figure of the apparent concavity of the ſky. In confirmation of which, I will conclude with Mr. *Cotes*'s account of a remarkable meteor (ſeen on the 6th of March $17\frac{1}{1}\frac{5}{6}$) written in anſwer to a letter which he received from Dr. *Dannye* then Rector of *Spofferth* in Yorkſhire. And confirmed by the appearance of comets tails and meteors.

170. The appearance of the meteor was very nearly the ſame with us here at *Cambridge* as with you, excepting that the triangular ſtreams of light were not ſo permanent as you ſeem to deſcribe them, and the point to which they all converged was diſtant from the zenith about 20 degrees, its azimuth lying between the ſouth and the eaſt at about 10 degrees from the ſouth, towards which point of the compaſs the wind tended. The poſition of this point of convergence may be more accurately determined, if there be occaſion: for at a quarter after ſeven when the appearance to us was in the greateſt perfection it lay nearly in the middle between the two bright ſtars in the heads of Caſtor and Pollux. I am told that ſome ſtreams were ſeen to ſhoot forth immediately after ſun ſet, and that they did not perfectly ceaſe till about 3 or 4 in the morning. The optical appearance of a remarkable meteor, ſolved by Mr. *Cotes*.

1 See *Scheiner*'s Refractiones cœleſtes ſive Solis Elliptici Phænomenon.

It was after ſeven before I had notice of this uncommon ſight. At firſt I ſaw only two or three of the triangular ſtreams towards the north and north weſt: theſe were not of long duration but were ſucceeded by others which appeared and vaniſhed again by turns, ariſing from and aſcending up to places in the heavens of very different altitudes above the horizon. From the time I began to view them they continued to aſcend more and more copiouſly, being propagated ſtill further and further from the north towards the weſt and eaſt and directed always to the heads of Gemini, till at length when they ſeemed almoſt to meet at the point of convergence, they began to aſcend up towards it from the ſouthern parts alſo and all around it; inſomuch that at a quarter after ſeven we had a perfect canopy of rays over us. The bottom of this canopy did no where reach down to the horizon; for near the north, where it deſcended the moſt, its altitude was about 10 or 15 degrees, and near the ſouth, where it deſcended the leaſt, its altitude was about 40 degrees. It remained in this ſtate about 2 minutes during which time we ſaw ſeveral colours ſome fainter and more permanent, others brighter, but quickly vaniſhing. Thus in the weſt I obſerved the rays to be tinged for ſome conſiderable time with an obſcure and heavy red: and in one of the brighteſt ſtreams at another time, there ſuddenly broke out a very vivid red which was inſtantly and gradually ſucceeded by the other priſmatick colours, all vaniſhing in about a ſecond of time. Theſe colours affected the ſenſe ſo ſtrongly that I thought them to be more intenſe than thoſe of the brighteſt rain-bow I had ever ſeen. A ſmall time before the appearance loſt its perfection, we were ſurprized to obſerve a ſhaking and trembling of the ſtreams chiefly in their upper parts, during which their convergence was confounded, and the whole heavens ſeemed to be in a convulſion. At the ſame time I could perceive waves of light towards the north which moved upwards and in their motion croſſed the ſtreams lying parallel to the horizon. Theſe waves were different from thoſe broad ones which you mention, and which I alſo took notice of: their breadth ſeemed to be about a degree, their length about 90 degrees; and I can compare them to nothing better than to thoſe ſlender waves upon the ſurface of ſtagnant water which are made by caſting in a ſmall ſtone.

Fig. 276. About ſeven or eight years ago, I happened to ſee a meteor which it will be of uſe to deſcribe to you. Along the horizon in the north there lay a white and luminous, and ſeemingly denſe matter in the form of a cloud repreſented by *abcd*; the length of it, *ab*, was about 10 or 15 degrees. From this there aroſe directly upwards pointed ſtreams of the like luminous and white matter which yet did not appear in any part of it to be ſo denſe as the former; and grew gradually more and more rare in its upper parts ſo as to vaniſh almoſt inſenſibly at the points. There was ſome little difference in the height of theſe ſtreams but they generally

aſcended

aſcended up to about 4 degrees above the horizon. They were very numerous and contiguous to each other, and ſeemed to be compoſed of very ſlender parallel filaments or rays. This was the common appearance, and the only remarkable thing which I farther obſerved was, that ſometimes a fire or flame would break out in the cloud, *abcd*, and move along it in a direction parallel to the horizon: and during this motion a pointed ſtream directly over the fire ſeemed to run along with it, and to paſs by the other more fixed ſtreams to which it always kept it ſelf parallel.

I am perſwaded that the late appearance was of the ſame kind with this which I have now been deſcribing. For let *AB* repreſent the plane of the horizon, *C* the place of the ſpectator, *EF* a fund of vapours or exhalations at a conſiderable height above us, diffuſed every way into a large and ſpacious plane parallel to the horizon. This fund of mixt matter by fermentation will emit ſtreams from it ſelf ſuch as *EG*, *FH*, &c. which if the wind be perfectly ſtill will aſcend perpendicularly upwards; if it be boiſterous and irregular they will be blended and confounded together, but if it be very gentle and uniform as it was at the time of our appearance they will be inclined towards the point of the horizon which is oppoſite to that from which the wind blows. Now if *ADB* repreſents the concave of the heavens, and a line *CD*, be drawn pararallel to the columns *EG*, *FH*, &c. it is certain by the rules of perſpective, that theſe columns will appear upon that concave to converge all around towards the point *D*: thus the column, *EG*, will ſeem to ariſe from the point *e*, to aſcend up to *g*, and to take up the ſpace *eg*: and in like manner the arch *fh*, will be the projection of the column *FH*. From hence it is evident that the reaſon why the triangular ſtreams aſcended at firſt only from the northern parts of the heavens was this: the fund of matter, *EF*, was not yet arrived by its motion to the line *CD*. After it had paſſed that line it is plain they muſt appear to aſcend from all quarters. A great number of columns being therefore diſpoſed to emit light at the ſame time cauſed that perfect canopy, which I deſcribed above. The reaſon why that canopy deſcended lower in the north, than in the ſouth was this: the ſhining columns which had not yet paſſed the line *CD* were more numerous and more remote from it than thoſe which had paſſed it, for if the point *E*, be farther diſtant from *CD* than the point *F*, the arch *Ae*, muſt needs be leſs than the arch *Bf*. An irregular guſt of wind blowing upon and ſhaking the columns (I ſuppoſe) was the cauſe of that trembling which appeared in the triangular ſtreams, and the cauſe alſo which deſtroyed that fine appearance of the canopy. The ſlender circular waves ſeen at the ſame time, might alſo be explained from the ſame cauſe. I need not detain you any longer, by endeavouring to make out ſome other particulars of this unuſual appearance: I fear I have been already

ready too tedious. However, I will not omit to mention a very eaſy contrivance by which the thing may be tolerably well repreſented to view. Take a hoop, and round about it faſten ſeveral ſtraight ſticks parallel to each other but all inclined to the plane of the hoop, hold this plane parallel to the horizon, and in that poſture move it with its ſticks over a candle, the ſhadow of the ſticks upon the ceiling of your room, will converge to a point, not directly over the candle (as they would have done had the ſticks been perpendicular to the plane of the hoop) but to the point in which a line drawn from the candle parallel to the ſticks, ſhall interſect the plane of the ceiling.

CHAP. VI.

CONCERNING THE ORIGINE AND CAUSE OF COLOURS.

Deſign.

To make this popular treatiſe more compleat, I have added Sir *Iſaac Newton*'s theory of colours, deſcribed in his own words as near as poſſible and proved by his own experiments.

I. Experiment. A deſcription of the ſun's image made by a priſm. Newt. Opt. p. 21. Fig. 278.

171. In a very dark chamber at a round hole *F*, about one third of an inch broad, made in the ſhut of a window, I placed a glaſs priſm *ABC* whereby the beam of the ſun's light *SF*, which came in at that hole, might be refracted upwards, toward the oppoſite wall of the chamber, and there form a coloured image of the ſun, repreſented at *PT*. The axis of the priſm, (that is the line paſſing through the middle of the priſm, from one end of it to the other end, parallel to the edge of the refracting angle) was in this and the following experiments perpendicular to the incident rays. About this axis I turned the priſm ſlowly, and ſaw the refracted light on the wall, or coloured image of the ſun, firſt to deſcend, and then to aſcend. Between the deſcent and aſcent when the image ſeemed ſtationary, I ſtopped the priſm and fixt it in that poſture.

Then I let the refracted light fall perpendicularly upon a ſheet of white paper, *MN*, placed at the oppoſite wall of the chamber, and obſerved the figure and dimenſions of the ſolar image, *PT*, formed on the paper by that light. This image was oblong and not oval, but terminated by two rectilinear and parallel ſides and two ſemicircular ends. On its ſides it was bounded pretty diſtinctly, but on its ends very confuſedly and indiſtinctly, the light there decaying and vaniſhing by degrees. At the diſtance of $18\frac{1}{2}$ feet from the priſm the breadth of the image was about $2\frac{1}{8}$ inches, but its length was about $10\frac{1}{4}$ inches, and the length of its rectilinear ſides about 8 inches; and *ACB* the refracting angle of the priſm, whereby ſo great a length was made, was 64 degrees. With a leſs angle the length of the image was leſs, the breadth remaining the ſame. It is farther to be obſerved that the rays went on in ſtraight lines from

from the priſm to the image, and therefore at their going out of the priſm had all that inclination to one another from which the length of the image proceeded. This image *PT* was coloured, and the more eminent colours lay in this order from the bottom at *T* to the top at *P*; red, orange, yellow, green, blue, indigo, violet; together with all their intermediate degrees in a continual ſucceſſion perpetually varying.

172. Our author concludes from this experiment, and many more to be mentioned hereafter, that the light of the ſun conſiſts of a mixture of ſeveral ſorts of coloured rays, ſome of which at equal incidences are more refracted than others, and therefore are called more refrangible. The red at *T*, being neareſt to the place *Y*, where the rays of the ſun would go directly if the priſm was taken away, is the leaſt refracted of all the rays; and the orange, yellow, green, blue, indigo and violet are continually more and more refracted, as they are more and more diverted from the courſe of the direct light. For by mathematical reaſoning he has proved, that when the priſm is fixt in the poſture above mentioned, ſo that the place of the image ſhall be the loweſt poſſible, or at the limit between its deſcent and aſcent, the figure of the image ought then to be round like the ſpot at *Y*, if all the rays that tended to it were equally refracted. Therefore ſeeing by experience it is found that this image is not round, but about 5 times longer than broad, it follows that all the rays are not equally refracted. And this concluſion is farther confirmed by the following experiments. Hence the ſun's rays are differently refrangible.

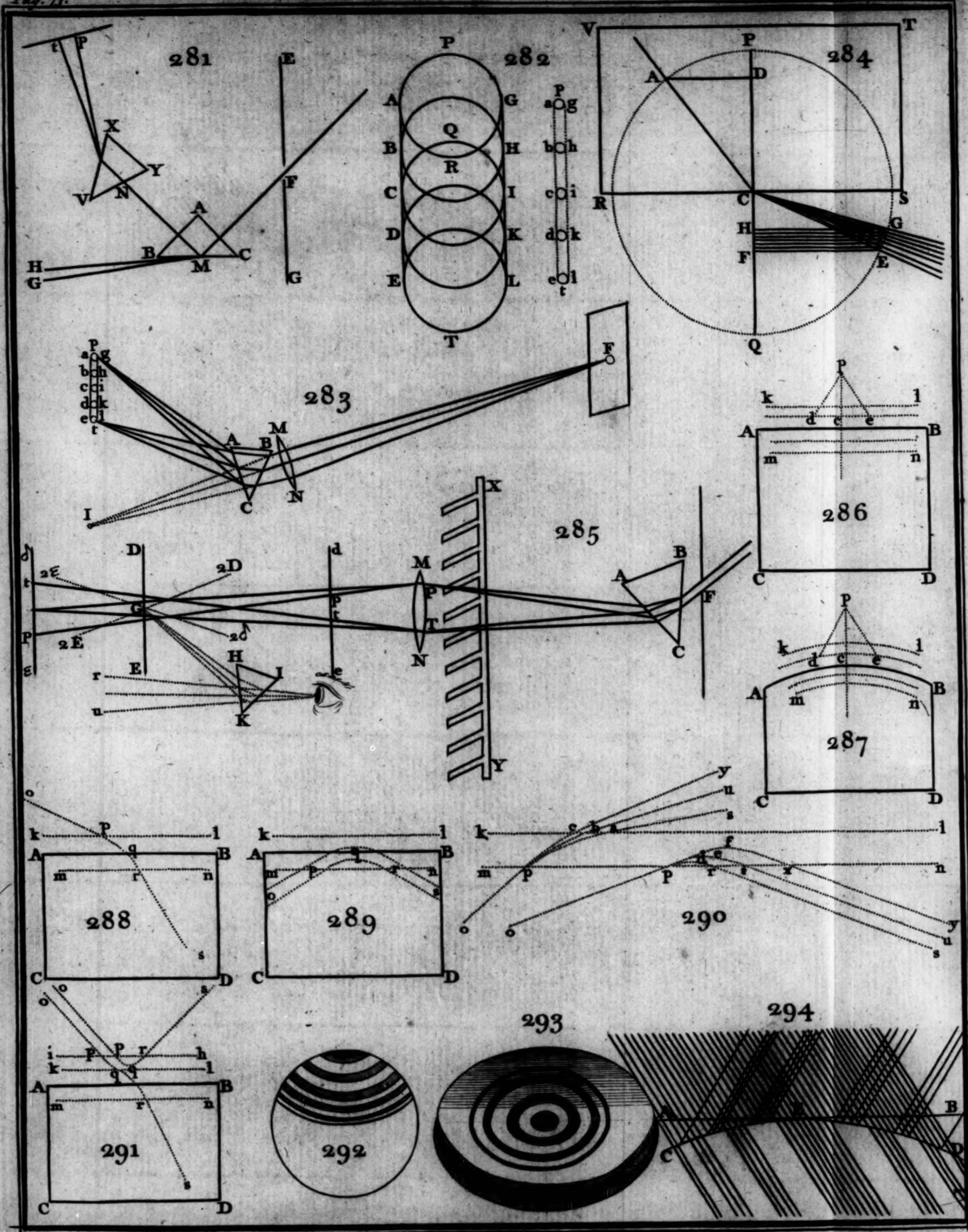
In the ſun beam *SF* which was propagated into the room through the hole in the window-ſhut *EG*, at the diſtance of ſome feet from the hole, I held the priſm *ABC* in ſuch a poſture, that its axis might be perpendicular to that beam: then I looked through the priſm upon the hole *F*, and turning the priſm to and fro about its axis to make the image *pt*, of the hole aſcend and deſcend, when between its two contrary motions it ſeemed ſtationary, I ſtopped the priſm; in this ſituation of the priſm viewing through it the ſaid hole *F*, I obſerved the length of its refracted image *pt* to be many times greater than its breadth; and that the moſt refracted part thereof appeared violet at *p*; the leaſt refracted red, at *t*; and the middle parts indigo, blue, green, yellow and orange in order. The ſame thing happened when I removed the priſm out of the ſun's light and looked through it upon the hole ſhining by the light of the clouds beyond it. And yet if the refractions of all the rays were equal according to one certain proportion of the ſines of incidence and refraction, as is vulgarly ſuppoſed, the refracted image ought to have appeared round, by the mathematical demonſtration above mentioned. So then by theſe two experiments it appears that in equal incidences there is a conſiderable inequality of refractions. II. Experiment. Newt. Opt. p. 27. Fig. 279.

For

For the discovery of this fundamental property of light, which has opened the whole mystery of colours, we see our author was not only beholden to the experiments themselves, which many others had made before him, but also to his skill in geometry; which was absolutely necessary to determine what the figure of the refracted image ought to be upon the old principle of an equal refraction of all the rays: but having thus made the discovery he contrived the following experiment to prove it at sight.

III. Experiment. Newt. Opt. p. 37. Fig. 280.

In the middle of two thin boards, *DE*, *de*, I made a round hole in each, at *G* and *g*, a third part of an inch in diameter; and in the window-shut a much larger hole being made, at *F*, to let into my darkened chamber a large beam of the sun's light, I placed a prism, *ABC*, behind the shut in that beam, to refract it towards the opposite wall; and close behind this prism I fixed one of the boards *DE*, in such manner that the middle of the refracted light might pass through the hole made in it at *G*, and the rest be intercepted by the board. Then at the distance of about 12 feet from the first board I fixed the other board *de*, in such manner that the middle of the refracted light, which came through the hole in the first board, and fell upon the opposite wall, might pass through the hole *g* in this other board *de*, and the rest being intercepted by the board might paint upon it the coloured spectrum of the sun. And close behind this board I fixed another prism *abc* to refract the light which came through the hole *g*. Then I returned speedily to the first prism *ABC* and by turning it slowly to and fro about its axis, I caused the image which fell upon the second board *de* to move up and down upon that board, that all its parts might pass successively through the hole in that board, and fall upon the prism behind it. And in the mean time I noted the places, *M*, *N*, on the opposite wall, to which that light after its refraction in the second prism did pass; and by the difference of the places at *M* and *N*, I found that the light, which being most refracted in the first prism *ABC*, did go to the blue end of the image, was again more refracted by the second prism *abc*, than the light which went to the red end of that image. For when the lower part of the light which fell upon the second board *de*, was cast through the hole *g*, it went to a lower place *M* on the wall; and when the higher part of that light was cast through the same hole *g*, it went to a higher place *N* on the wall; and when any intermediate part of the light was cast through that hole, it went to some place in the wall between *M* and *N*. The unchanged position of the holes in the boards made the incidence of the rays upon the second prism to be the same in all cases. And yet in that common incidence some of the rays were more refracted and others less: and those were more refracted in this prism, which by a greater refraction in the first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first prism were more turned out of their way; and therefore for their constancy of being more refracted are deservedly called more refrangible.

Our author shews also, by experiments made with a convex glass, that lights (reflected from natural bodies) which differ in colour, differ also in degrees of refrangibility [a]: and that they differ in the same manner as the rays of the sun do.

a Newt. Opt. p. 16.

IV. Experiment. Rays of the sun are differently reflexible. Newt. Opt. p. 45. Fig. 281.

173. The sun's light consists of rays differing in reflexibility, and those rays are more reflexible than others which are more refrangible. A prism *ABC* whose two angles at its base *BC* were equal to one another and half right ones, and the third at *A* a right one, I placed in a beam *FM* of the sun's light, let into a dark chamber through a hole *F* one third part of an inch broad. And turning the prism slowly about its axis until the light which went through one of its angles *ACB* and was refracted by it to *G* and *H*, began to be reflected into the line *MN* by its base *BC*, at which till then it went out of the glass; I observed that those rays as *MH* which had suffered the greatest refraction were sooner reflected than the rest. To make it evident that the rays which vanished at *H* were reflected into the beam *MN*, I made this beam pass through another prism *VXY*, and being refracted by it to fall afterwards upon a sheet of white paper *pt* placed at some distance behind it and there by that refraction to paint the usual colours at *pt*. Then causing the first prism to be turned about its axis according to the order of the letters *ABC*, I observed that when those rays *MH* which in this prism had suffered the greatest refraction and appeared blue and violet, began to be totally reflected, the blue and violet light on the paper which was most refracted in the second prism received a sensible increase at *p*, above that of the red and yellow at *t*: and afterwards when the rest of the light, which was green yellow and red began to be totally reflected and vanished at *G*, the light of those colours at *t* on the paper *pt* received as great an increase as the violet and blue had received before. Which puts it past dispute that those rays became first of all totally reflected at the base *BC*, which before at equal incidences with the rest upon the base *BC* had suffered the greatest refraction. I do not here take notice of any refractions made in the sides *AC*, *AB* of the first prism, because the light enters almost perpendicularly at the first side and goes out almost perpendicularly at the second, and therefore suffers none, or so little that the angles of incidence at the base *BC* are not sensibly altered by it; especially if the angles of the prism at the base *BC* be each about 40 degrees. For the rays *FM* begin to be totally reflected when the angle *CMF* is about 50 degrees [b], and therefore they will then make a right angle of 90 degrees with *AC*.

b Art. 17.

It appears also from this experiment that the beam of light *MN*, reflected by the base of the prism, being augmented first by the more re-

 frangible

frangible rays and afterwards by the less refrangible is composed of rays differently refrangible.

Definitions Newt. Opt. p. 4. The light whose rays are all alike refrangible I call simple homogeneal and similar, and that whose rays are some more refrangible than others I call compound, heterogeneal and dissimilar. The former light I call homogeneal not because I would affirm it so in all respects; but because the rays which agree in refrangibility agree at least in all their other properties which I consider in the following discourse.

The colours of homogeneal lights I call primary, homogeneal and simple, and those of heterogeneal lights, heterogeneal and compound. For these are always compounded of homogeneal lights, as will appear in the following discourse.

Newt. Opt. p. 108. The homogeneal light and rays which appear red, or rather make objects appear so, I call rubrifick or red-making; those which make objects appear yellow, green, blue and violet, I call yellow-making, green-making, blue-making, violet-making; and so the rest. And if at any time I speak of light and rays as coloured or endued with colours, I would be understood to speak not philosophically and properly but grosly, and according to such conceptions as vulgar people in seeing all these experiments would be apt to frame. For the rays to speak properly are not coloured. In them there is nothing else than a certain power and disposition to stir up a sensation of this or that colour. For as sound in a bell or musical string or other sounding body, is nothing but a trembling motion, and in the air nothing but that motion propagated from the object, and in the sensorium it is a sense of that motion under the form of sound; so colours in the object are nothing but a disposition to reflect this or that sort of rays more copiously than the rest; in the rays they are nothing but their dispositions to propagate this or that motion into the sensorium; and in the sensorium they are sensations of those motions under the forms of colours.

How the long image is composed of circles of different sorts of rays. Opt. p. 31. Fig. 282.

174. By the mathematical proposition above mentioned [a], it is certain that the rays which are equally refrangible do fall upon a circle answering to the sun's apparent disque, which will also be proved by experiment by and by. Now let *AG* represent the circle which all the most refrangible rays, propagated from the whole disque of the sun, would illuminate and paint upon the opposite wall if they were alone; *EL* the circle which all the least refrangible rays would in like manner illuminate if they were alone; *BH*, *CI*, *DK* the circles which so many intermediate sorts would paint upon the wall, if they were singly propagated from the sun in successive order, the rest being intercepted; and conceive that there are other circles without number, which innumerable other intermediate sorts of rays would successively paint upon the wall, if the sun should successively emit every sort apart. And seeing the sun emits

[a] Art. 172.

emits all these sorts at once, they must all together illuminate and paint innumerable equal circles; of all which, being according to their degrees of refrangibility placed in order in a continual series, that oblong spectrum *PT* is composed, which was described in the first experiment.

175. Now if these circles whilst their centers keep their distances and positions could be made less in diameter, their interfering one with another and consequently the mixture of the heterogeneous rays would be proportionably diminished. Let the circles *AG*, *BH*, *CI*, &c. remain as before; and let *ag*, *bh*, *ci*, &c. be so many less circles lying in a like continual series, between two parallel right lines *ae* and *gl*, with the same distances between their centers, and illuminated with the same sorts of rays: that is, the circle *ag* with the same sort by which the corresponding circle *AG* was illuminated; and the rest of the circles *bh*, *ci*, *dk*, *el* respectively with the same sorts of rays by which the corresponding circles *BH*, *CI*, *DK*, *EL*, were illuminated. In the figure *PT* composed of the great circles, three of those circles *AG*, *BH*, *CI* are so expanded into each other, that three sorts of rays, by which those circles are illuminated, together with innumerable other sorts of intermediate rays, are mixed at *QR* in the middle of the circle *BH*. And the like mixture happens throughout almost the whole length of the figure *PT*. But in the figure *pt*, composed of the less circles, the three less circles *ag*, *bh*, *ci*, which answer to those three greater do not extend into one another; nor are there any where mingled so much as any two of the three sorts of rays by which those circles are illuminated, and which in the figure *PT* are all of them intermingled at *QR*. So then if we would diminish the mixture of the rays we are to diminish the diameters of the circles. Now these would be diminished if the sun's diameter, to which they answer, could be made less than it is, or (which comes to the same purpose) if without doors, at a great distance from the prism towards the sun, some opake body were placed with a round hole in the middle of it to intercept all the sun's light, except so much as coming from the middle of his body could pass through that hole to the prism. For so the circles *AG*, *BH* and the rest, would not any longer answer to the whole disque of the sun, but only to that part of it which could be seen from the prism through that hole; that is to the apparent magnitude of that hole viewed from the prism. But that these circles may answer more distinctly to that hole, a lens is to be placed by the prism to cast the image of the hole, (that is every one of the circles *AG*, *BH*, &c.) distinctly upon the paper at *PT*; after such a manner as by a lens placed at a window the pictures of objects abroad are cast distinctly upon a paper within the room. If this be done it will not be necessary to place that hole very far off, no not beyond the window. And therefore instead of that hole I used the hole in the window-shut as follows.

How those sorts of rays may be farther separated. Newt. Opt. p. 54. Fig. 282.

V. Experiment. Newt. Opt. p. 57. Fig. 283.

In the ſun's light let into my darkned chamber through a ſmall round hole in my window-ſhut, at about ten or twelve feet from the window, I placed a lens *MN*, by which the image of the hole *F*, might be diſtinctly caſt upon a ſheet of white paper placed at *I*. Then immediately after the lens I placed a priſm *ABC*, by which the trajected light might be refracted either upwards or ſideways, and thereby the round image which the lens alone did caſt upon the paper at *I*, might be drawn out into a long one with parallel ſides as repreſented at *pt*. This oblong image I let fall upon another paper at about the ſame diſtance from the priſm as the image at *I*, moving the paper either towards the priſm or from it, untill I found the juſt diſtance where the rectilinear ſides of the image *pt* became moſt diſtinct. For in this caſe the circular images of the hole, which compoſe that image, after the manner that the circles *ag*, *bh*, *ci*, &c. do the figure *pt*, were terminated moſt diſtinctly, and therefore extended into one another the leaſt that they could, and by conſequence the mixture of the heterogeneous rays was now the leaſt of all. The circles, *ag*, *bh*, *ci*, &c, which compoſe the image *pt*, are each equal to the circle at *I*, and therefore, by diminiſhing the hole *F* or by removing the lens farther from it, may be diminiſhed at pleaſure, whilſt their centers keep the ſame diſtances from each other. Thus by diminiſhing the breadth of the image *pt* the circles of heterogeneal rays that compoſe it, may be ſeparated from each other as much as you pleaſe. Yet inſtead of the circular hole *F*, it is better to ſubſtitute an oblong hole ſhaped like a parallelogram with its length parallel to the length of the priſm. For if this hole be an inch or two long and but a tenth or twentieth part of an inch broad, or narrower, the light of the image *pt* will be as ſimple as before or ſimpler; and the image being much broader is therefore fitter to have experiments tried in its light than before.

VI. Experiment. Homogeneal light is refracted regularly &c. Newt. Opt. p. 62.

176. Homogeneal light is refracted regularly without any dilatation ſplitting or ſhattering of the rays, and the confuſed viſion of objects ſeen through refracting bodies by heterogeneal light, ariſes from the different refrangibility of ſeveral ſorts of rays. This will appear by the experiments which follow. In the middle of a black paper I made a round hole about a fifth or a ſixth part of an inch in diameter. Upon this paper I cauſed the ſpectrum of homogeneal light deſcribed in the former article, ſo to fall that ſome part of the light might paſs through the hole in the paper. This tranſmitted part of the light I refracted with a priſm placed behind the paper, and letting this refracted light fall perpendicularly upon a white paper two or three feet diſtant from the priſm, I found that the ſpectrum formed on the paper by this light was not oblong, as when it is made, in the firſt experiment, by refracting the ſun's compound light, but was (ſo far as I could judge by my eye) perfectly circular, the length being no where greater than the breadth; which ſhews that this light

light is refracted regularly without any dilatation of the rays; and is an ocular demonstration of the mathematical proposition mentioned in the 172d article.

In the homogeneal light I placed a paper circle of a quarter of an inch in diameter; and in the sun's unrefracted, heterogeneal, white light I placed another paper circle of the same bigness; and going from these papers to the distance of some feet I viewed both circles through a prism. The circle illuminated by the sun's heterogeneal light appeared very oblong as in the 2d experiment, the length being many times greater than the breadth. But the other circle illuminated with homogeneal light appeared circular and distinctly defined, as when it is viewed by the naked eye; which proves the whole proposition mentioned at the beginning of this article.

VII. Experiment. Newt. Opt. p. 63.

In the homogeneal light I placed flies and such like minute objects, and viewing them through a prism I saw their parts as distinctly defined as if I had viewed them with the naked eye. The same objects placed in the sun's unrefracted heterogeneal light which was white, I viewed also through a prism, and saw them most confusedly defined, so that I could not distinguish their smaller parts from one another. I placed also the letters of a small print one while in the homogeneal light and then in the heterogeneal, and viewing them through a prism they appeared in the latter case so confused and indistinct that I could not read them; but in the former they appeared so distinct that I could read readily, and thought I saw them as distinct as when I viewed them with my naked eye; in both cases I viewed the same objects through the same prism at the same distance from me and in the same situation. There was no difference but in the lights by which the objects were illuminated and which in one case was simple in the other compound; and therefore the distinct vision in the former case and confused in the latter could arise from nothing else than from that difference in the lights. Which proves the whole proposition.

VIII. Experiment. Ibid.

177. In these three experiments it is farther very remarkable that the colour of homogeneal light was never changed by the refraction: and as these colours were not changed by refractions, so neither were they by reflexions. For all white, grey, red, yellow, green, blue, violet bodies, as paper, ashes, red lead, orpiment, indigo, bise, gold, silver, copper, grass, blue flowers, violets, bubbles of water tinged with various colours, peacocks feathers, the tincture of *lignum nephriticum* and such like, in red homogeneal light appeared totally red, in blue light totally blue, in green light totally green, and so of other colours. In the homogeneal light of any colour they all appeared totally of that same colour, with this only difference, that some of them reflected that light more strongly, others

The colour of homogeneal light cannot be changed by refractions nor by reflexions. Newt. Opt. p. 107.

others more faintly. I never yet found any body which by reflecting homogeneal light could sensibly change its colour.

From all which it is manifest, that if the sun's light consisted of but one sort of rays, there would be but one colour in the world. Nor would it be possible to produce any new colour by reflexions and refractions: and by consequence that the variety of colours depends upon the composition of light.

All homog. light has its proper colour answering to its degree of refrangibility. Newt. Opt. p. 106.

178. The solar image *pt* formed by the separated rays in the 5th experiment, did in the progress from its end *p*, on which the most refrangible rays fell, unto its end *t*, on which the least refrangible rays fell, appear tinged with this series of colours; violet, indigo, blue, green, yellow, orange, red, together with all their intermediate degrees in a continual succession perpetually varying: so that there appeared as many degrees of colours as there were sorts of rays differing in refrangibility. And since these colours could not be changed by refractions nor by reflexions, it follows that all homogeneal light has its proper colour answering to its degree of refrangibility.

The sine of incidence of every homog. ray is to its sine of refraction in a given ratio. Newt. Opt. p. 64. Fig. 284. [a] Art. 13.

179. Every homogeneal ray considered apart is refracted according to one and the same rule[a], so that its sine of incidence is to its sine of refraction in a given ratio: that is, every different coloured ray has a different ratio belonging to it. This our author has proved by experiment, and by other experiments has determined by what numbers those given ratios are expressed. For instance, if an heterogeneal white ray of the sun emerges out of glass into air, or which is the same thing, if rays of all colours be supposed to succeed one another in the same line *AC*, and *AD* their common sine of incidence in glass be divided into 50 equal parts, then *EF* and *GH* the sines of refraction into air, of the least and most refrangible rays will be 77 and 78 such parts respectively. And since every colour has several degrees, the sines of refraction of all the degrees of red will have all intermediate degrees of magnitude from 77 to $77\frac{1}{8}$, of all the degrees of orange from $77\frac{1}{8}$ to $77\frac{1}{5}$, of yellow from $77\frac{1}{5}$ to $77\frac{1}{3}$, of green from $77\frac{1}{3}$ to $77\frac{1}{2}$, of blue from $77\frac{1}{2}$ to $77\frac{2}{3}$, of indigo from $77\frac{2}{3}$ to $77\frac{7}{9}$, and of violet from $77\frac{7}{9}$ to 78[b].

[b] Newt. Opt. p. 109.

The different properties of simple and compound colours. Newt. Opt. p. 115.

180. Colours may be produced by composition which shall be like to the colours of homogeneal light, as to the appearance of colour, but not as to the immutability of colour and constitution of light. And those colours, by how much they are more compounded, by so much are they less full and intense; and by too much composition they may be diluted and weakened till they cease, and the mixture becomes white or grey. There may be also colours produced by composition, which are not fully like any of the colours of homogeneal light. For a mixture of homogeneal red and yellow compounds an orange, like in appearance of colour to that orange which in the series of unmixed prismatick colours lyes

between

between them. But the light of one orange is homogeneal as to refrangibility, that of the other is heterogeneal; and the colour of the one, if viewed through a prism remains unchanged, that of the other is changed and resolved into its component colours red and yellow. And after the same manner other neighbouring homogeneal colours may compound new colours, like the intermediate homogeneal ones: as yellow and green the colour between them both; and afterwards if blue be added there will be made a green, the middle colour of the three which enter the composition. For the yellow and blue on either hand, if they are equal in quantity, draw the intermediate green equally towards themselves, and so keep it as it were in æquilibrio, that it verge not more to the yellow on one hand, than to the blue on the other, but by their mixed actions remain still a middle colour. To this mixed green there may be farther added some red and violet, and yet the green will not presently cease but only grow less full and vivid; and by increasing the red and violet, it will grow more and more dilute, untill by the prevalence of the added colours it be overcome and turned into whiteness or some other colour. So if to the colour of any homogeneal light, the sun's white light composed of all sorts of rays be added, that colour will not vanish or change its species, but be diluted, and by adding more and more white it will be diluted more and more perpetually. Lastly if red and violet be mingled there will be generated according to their various proportions various purples: such as are not like in appearance to the colour of any homogeneal light; and of these purples mixed with yellow and blue may be made other new colours.

IX. Experiment. Whiteness may be compounded of colours. Newt. Opt. p. 117. Fig. 285.

181. Whiteness and all grey colours between white and black, may be compounded of colours; and the whiteness of the sun's light is compounded of all the primary colours mixed in a due proportion.

For let the solar image *PT* fall upon a lens *MN* above four inches broad and about six feet distant from the prism *ABC*, and so figured that it may cause the coloured light which divergeth from the prism to converge and meet again at its focus *G* about 6 or 8 feet distant from the lens, and there to fall perpendicular upon a white paper *DE*. And if you move this paper to and fro, you will perceive that near the lens, as at *de*, the whole solar image, suppose at *pt*, will appear upon it intensely coloured after the manner above explained: and that by receding from the lens those colours will perpetually come towards one another, and by mixing more and more dilute one another continually, untill at length the paper comes to the focus *G*, where by a perfect mixture they will wholly vanish and be converted into whiteness, the whole light appearing now upon the paper like a little white circle. And afterwards by receding farther from the lens, the rays which before converged will now cross one another in the focus *G*, and diverge from thence and thereby make

make the colours to appear again, but yet in a contrary order, ſuppoſe at *δε*, where the red *t* is now above which before was below, and the violet *p* is below which before was above.

Let us now ſtop the paper at the focus *G* where the light appears totally white and circular, and let us conſider its whiteneſs. I ſay that this is compoſed of all the converging colours. For if any of thoſe colours be intercepted at the lens, the whiteneſs will ceaſe, and degenerate into that colour which ariſeth from the compoſition of the other colours which are not intercepted. And then if the intercepted colours be let paſs and fall upon that compound colour, they mix with it and by their mixture reſtore the whiteneſs. So if the violet, blue and green be intercepted, the remaining yellow, orange and red will compound upon the paper an orange, and then if the intercepted colours be let paſs, they will fall upon this compound orange and together with it decompound a white. So alſo if the red and violet be intercepted the remaining yellow, green and blue will compound a green upon the paper, and the red and violet being let paſs will fall upon this green, and together with it will decompound a white. And that in this compoſition of white, the ſeveral rays do not ſuffer any change in their colorifick qualities by acting upon one another, but are only mixed, and by a mixture of their colours produce white, may farther appear by theſe arguments.

If the paper be placed beyond the focus *G*, ſuppoſe at *δε*, and then the red colour at the lens be alternately intercepted and let paſs again, the violet colour on the paper will not ſuffer any change thereby, as it ought to do if the ſeveral ſorts of rays acted upon one another in the focus *G*, where they croſs. Neither will the red upon the paper be changed by any alternate ſtopping and letting paſs the violet which croſſes it.

And if the paper be placed at the focus *G* and the white round image at *G* be viewed through the priſm *HIK*, and by the refraction of that priſm be tranſlated to the place *rv* and there appear tinged with various colours; namely the violet at *v* and red at *r*, and others between; and then the red colour at the lens be often ſtopped and let paſs by turns, the red at *r* will accordingly diſappear and return as often; but the violet at *v* will not thereby ſuffer any change. And ſo by ſtopping and letting paſs alternately the blue at the lens, the blue at *v* will accordingly diſappear and return without any change made in the red at *r*. The red therefore depends on one ſort of rays and the blue on another ſort, which in the focus *G* where they were commixed do not act on one another. And there is the ſame reaſon of the other colours.

I conſidered farther that when the moſt refrangible rays *Pp* and the leaſt refrangible *Tt* are by converging inclined to one another, the paper if held very oblique to thoſe rays in the focus *G*, might reflect one ſort of them more copiouſly than the other ſort, and by that means the reflect-

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ed light would be tinged in that focus with the colour of the predominant rays, provided thoſe rays ſeverally retained their colours or colorific qualities in the compoſition of white. But if they did not retain them in that white, but became all of them ſeverally endued there with a diſpoſition to ſtrike the ſenſe with the perception of white, then they could never loſe their whiteneſs by ſuch reflections. I inclined therefore the paper to the rays very obliquely, as repreſented in the poſture 2δ, 2ϵ, that the moſt refrangible rays *Pp*, falling more directly and conſequently more denſely upon it than the reſt, might be more copiouſly reflected than the reſt, and the whiteneſs at length changed ſucceſſively into blue, indigo and violet. Then I inclined it the contrary way as repreſented in the poſture 2*D*, 2*E*, that the leaſt refrangible rays *Tt* might fall more directly upon it, and conſequently be more copious in the reflected light than the reſt, and the whiteneſs turned ſucceſſively to yellow, orange and red.

Laſtly I made an inſtrument *XY* in faſhion of a comb whoſe teeth being in number 16 were about an inch and a half broad, and the intervals of the teeth about 2 inches wide. Then by interpoſing ſucceſſively the teeth of this inſtrument near the lens, I intercepted part of the colours by the interpoſed tooth, whilſt the reſt of them went on through the interval of the teeth to the paper *DE*, and there painted a round ſolar image. But the paper I had placed ſo that the image might appear white as often as the comb was taken away; and then the comb being interpoſed as was ſaid, that whiteneſs, by reaſon of the intercepted part of the colours at the lens, did always change into the colour compounded of thoſe colours which were not intercepted; and that colour was by the motion of the comb perpetually varied ſo, that in the paſſing of every tooth over the lens all theſe colours red, yellow, blue, green, purple did always ſucceed one another. I cauſed therefore all the teeth to paſs ſucceſſively over the lens, and when the motion was ſlow, there appeared a perpetual ſucceſſion of the colours upon the paper, but if I ſo accelerated the motion that the colours by reaſon of their quickneſs, could not be diſtinguiſhed from one another, the appearance of the ſingle colours ceaſed. There was no red, no yellow, no green, no blue, nor purple to be ſeen any longer, but from a confuſion of them all there aroſe one uniform white colour. Of the light which now by the mixture of all the colours appeared white, there was no part really white. One part was red another yellow, a third green, a fourth blue, a fifth purple, and every part retains its proper colour till it ſtrikes the ſenſorium. If the impreſſions follow one another ſlowly, ſo that they may be ſeverally perceived, there is made a diſtinct ſenſation of all the colours one after another in a continual ſucceſſion. But if the impreſſions follow one another ſo quick that they cannot be ſeverally perceived, there ariſeth

out of them all one common ſenſation, which is neither of this colour alone, nor of that alone, but hath it ſelf indifferently to them all, and this is a ſenſation of whiteneſs. By the quickneſs of the ſucceſſions the impreſſions of the ſeveral colours are confounded in the ſenſorium, and out of that confuſion ariſeth a mixt ſenſation. If a burning coal be nimbly moved round in a circle with gyrations continually repeated, the whole circle will appear like fire. The reaſon of which is, that the ſenſation of the coal in the ſeveral places of that circle remains impreſſed on the ſenſorium till the coal returns again to the ſame place. And ſo in a quick conſecution of colours the impreſſion of every colour remains in the ſenſorium, until a revolution of all the colours be compleated and that firſt colour returns again. The impreſſions therefore of all the colours are at once in the ſenſorium and jointly ſtir up a ſenſation of them all: and ſo it is manifeſt from this experiment that the commixed impreſſions of all the colours do ſtir up and beget a ſenſation of white, that is, that whiteneſs is compounded of all the colours.

X. Experiment. Newt. Opt. p. 129.

Hitherto I have produced whiteneſs by mixing the priſmatick colours. If now the colours of natural bodies are to be mingled, let a little water thickened with ſoap be agitated to raiſe a froth, and after that froth has ſtood a little, there will appear to one that ſhall view it intently various colours every where in the ſurface of the ſeveral bubbles, but to one that ſhall go ſo far off that he cannot diſtinguiſh the colours from one another, the whole froth will grow white with a perfect whiteneſs.

XI. Experiment. Ibid.

Laſtly in attempting to compound a white by mixing the coloured powders which painters uſe, I conſidered that all coloured powders do ſuppreſs and ſtop in them a very conſiderable part of the light by which they are illuminated. For they become coloured by reflecting the light of their own colour more copiouſly, and that of all other colours more ſparingly; and yet they do not reflect the light of their own colours ſo copiouſly as white bodies do. If red lead, for inſtance, and a white paper be placed in the red light of the coloured ſpectrum made in the dark chamber by the refraction of a priſm, as is deſcribed in the 5th experiment, the paper will appear more lucid than the red lead, and therefore reflects the red-making rays more copiouſly than red lead doth. And if they be held in the light of any other colour, the light reflected by the paper will exceed the light reflected by the red lead in a much greater proportion. And the like happens in powders of other colours. And therefore by mixing ſuch powders we are not to expect a ſtrong and full white, ſuch as is that of paper, but ſome duſky obſcure one, ſuch as might ariſe from a mixture of light and darkneſs, or from white and black, that is a grey or dun, or ruſſet brown, ſuch as are the colours of a man's nails, of a mouſe, of aſhes, of ordinary ſtones, of mortar, of duſt and dirt in the high-ways, and the like; and ſuch a white I have of-

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ten produced by mixing coloured powders. For thus one part of red lead, and five parts of *viride æris* composed a dun colour like that of a mouse. For these two colours were severally so compounded of others, that in both together were a mixture of all colours; and there was less red lead than *viride æris* because of the fulness of its colour. Again, one part of red lead and four parts of blue bise, composed a dun colour verging a little to purple, and by adding to this a certain mixture of orpiment and *viride æris* in due proportion, the mixture lost its purple tincture and became perfectly dun. But the experiment succeeded better without minium, thus. To orpiment I added by little and little a certain full bright purple, which painters use, until the orpiment ceased to be yellow, and became a pale red. Then I diluted that red by adding a little *viride æris* and a little more blue bise than *viride æris*, until it became of such a grey or pale white as verged to no one of the colours more than another: for thus it became of a colour equal in whiteness to that of ashes or of wood newly cut or of a man's skin. The orpiment reflected more light than did any other of the powders, and therefore conduced more to the whiteness of the compound colour than they. To assign the proportions accurately may be difficult by reason of the different goodness of powders of the same kind. According as the colour of any powder is more or less full or luminous, it ought to be used in a less or greater proportion.

Now considering that these grey and dun colours may be also produced by mixing whites and blacks, and by consequence differ from perfect whites, not in species of colours but only in degrees of luminousness, it is manifest there is nothing more requisite to make them perfect whites than to increase their light sufficiently: and on the contrary if by increasing their light they can be brought to perfect whiteness, it will thence also follow that they are of the same species of colour with the best whites, and differ from them only in quantity of light, and this I tryed as follows. I took the third of the above mentioned grey mixtures (that which was compounded of orpiment, purple, bise and *viride æris*) and rubbed it thickly upon the floor of my chamber, where the sun shone upon it through the open casement; and by it, in the shadow, I placed a piece of white paper of the same bigness. Then going from them to the distance of 12 or 18 feet, so that I could not discern the unevenness of the surface of the powder; nor the little shadows let fall from the gritty particles thereof; the powder appeared intensely white so as to transcend even the paper it self in whiteness, especially if the paper were a little shaded from the light of the clouds; and then the paper compared with the powder appeared of such a grey colour as the powder had done before. But by laying the paper where the sun shines through the glass window, or by shutting the casement, that the sun might shine through

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glass upon the powder, and by using such other fit means of increasing or decreasing the lights wherewith the powder and paper were illuminated, the light wherewith the powder is illuminated may be made stronger, in such a due proportion, than the light wherewith the paper is illuminated, that they shall both appear exactly alike in whiteness. Now if you consider that this white of the powder in the sun-shine was compounded of the colours which the component powders have in the sunshine, you must acknowledge by this experiment as well as by the former that perfect whiteness may be compounded of colours.

The permanent colours of natural bodies explained.

182. The colours of natural bodies arise from hence, that some of them reflect some sorts of rays, others other sorts more copiously than the rest. Minium reflects the least refrangible or red-making rays most copiously and thence appears red. Violets reflect the most refrangible most copiously, and thence have their colour: and so of other bodies. Every body reflects the rays of its own colour more copiously than the rest, and from their excess and predominance in the reflected light has its colour.

XII. Experiment. Newt. Opt. p. 157.

For if in the homogeneal lights obtained by the 5th experiment, you place bodies of several colours, you will find as I have done, that every body looks more splendid and luminous in the light of its own colour. Cinnaber in the homogeneal red is most resplendent, in the green light it is manifestly less resplendent, in the blue light still less. Indigo in the violet blue light is most resplendent, and its splendor is gradually diminished as it is removed thence by degrees through the green and yellow light to the red. By a leek the green light, and next that the blue and yellow which compound green, are more strongly reflected than the other colours red and violet, and so of the rest. But to make these experiments the more manifest, such bodies ought to be chosen as have the fullest and most vivid colours, and two of those bodies are to be compared together. Thus for instance, if cinnaber and ultra-marine blue, or some other full blue be held together in the red homogeneal light, they will both appear red; but the cinnaber will appear of strongly luminous and resplendent red, and the ultra-marine blue of a faint obscure and dark red. And if they be held together in the blue homogeneal light, they will both appear blue; but the ultra-marine will appear of a strongly luminous and resplent blue, and the cinnaber of a faint and dark blue. Which puts it out of dispute that the cinnaber reflects the red light much more copiously than the ultra-marine doth, and the ultra-marine reflects the blue light much more copiously than the cinnaber doth. The same experiment may be tried successively with red and indigo or with any other two coloured bodies, if due allowance be made for the different strength or weakness of their colour and light.

And

And that this is not only a true reaſon of their colours, but even the only reaſon, may appear farther from this conſideration; that the colour of homogeneal light cannot be changed by the reflection of natural bodies. For if bodies by reflection cannot in the leaſt change the colour of any one ſort of rays, they cannot appear coloured by any other means, than by reflecting thoſe which either are of their own colour, or by mixture muſt produce it.

In tranſparent coloured liquors it is obſervable that their colour uſes to vary with their thickneſs. Thus for inſtance, a red liquor in a conical glaſs held between the light and the eye, looks of a pale and dilute yellow at the bottom where it is thin; and a little higher, where it is thicker grows orange; and where it is ſtill thicker becomes red; and where it is thickeſt the red is deepeſt and darkeſt. For it is to be conceived that ſuch a liquor ſtops the indigo-making, and violet making rays moſt eaſily, the blue-making rays more difficultly, the green-making rays ſtill more difficultly, and the red-making moſt difficultly: and that if the thickneſs of the liquor be only ſuch as ſuffices to ſtop a competent number of the violet-making and indigo-making rays, without diminiſhing much the number of the reſt, the reſt muſt compound a pale yellow; the colour in the ſolar image that lyes in the middle of them, as may be tryed by ſtopping the violet and indigo at the lens in the 9th experiment, and letting the reſt paſs on to be mixed together at the focus. But if the liquor be of ſuch a thickneſs as to ſtop alſo a great number of the blue-making rays and ſome of the green-making, the reſt muſt compound an orange. And where it is ſo thick as to ſtop alſo a great number of the green-making and a conſiderable number of the yellow-making, the reſt muſt begin to compound a red; and this red muſt grow deeper and darker as the yellow-making and orange-making rays are more and more ſtopped by increaſing the thickneſs of the liquor, ſo that few rays beſides the red-making can get through.

If there be two liquors of full colours ſuppoſe a red and a blue, and both of them ſo thick, as ſuffices to make their colours ſufficiently full; though either liquor be ſufficiently tranſparent apart, yet will you not be able to ſee through both together. For if only the red-making rays paſs through one liquor, and only the blue-making through the other, no rays can paſs through both. This Mr. *Hook* tried caſually with glaſs wedges filled with red and blue liquors, and was ſurprized at the unexpected event, the reaſon of it being then unknown [1].

Now whilſt bodies become coloured by reflecting or tranſmitting this or that ſort of rays more copiouſly than the reſt, it is to be conceived that they ſtop and ſtifle in themſelves the rays which they do not reflect or tranſmit. For if gold be foliated and held between your eye and the

1. *Hook*'s Micrography p. 73.

light,

light, the light looks of a greeniſh blue; and therefore maſſy gold lets into its body the blue-making rays to be reflected to and fro within it till they be ſtopped and ſtifled; whilſt it reflects the yellow-making outwards and thereby looks yellow. And much after the ſame manner that leaf gold is yellow by reflected and blue by tranſmitted light, there are ſome ſorts of liquors, as the tincture of *lignum nephriticum*, and ſome ſorts of glaſs, which tranſmit one ſort of light moſt copiouſly and reflect another ſort, and thereby look of ſeveral colours according to the poſition of the eye to the light. A tranſparent body which looks of any colour by tranſmitted light, may alſo look of the ſame colour by reflected light, the light of that colour being reflected by the farther ſurface of the body [a].

a Art. 17.

C H A P. VII.

CONCERNING THE CAUSE OF REFRACTION, REFLECTION, INFLECTION AND EMISSION OF LIGHT.

Reflexion not cauſed by the impinging of light upon the medium. Newt. Opt. p. 237.

183. THAT the cauſe of reflection is not the impinging of light on the ſolid or impervious parts of bodies, as is commonly believed, will appear by the following conſiderations. Firſt that in the paſſage of light out of glaſs into air there is a reflection as ſtrong as in its paſſage out of air into glaſs, or rather a little ſtronger, and by many degrees ſtronger than in its paſſage out of glaſs into water. And it ſeems not probable that air ſhould have more reflecting parts than water or glaſs. But if that ſhould poſſibly be ſuppoſed, yet it will avail nothing; for the reflection is as ſtrong or ſtronger when the air is drawn away from the glaſs, by an air-pump, as when it is adjacent to it. Secondly, if light in its paſſage out of glaſs into air be incident more obliquely than an angle of 40 or 41 degrees, it is wholly reflected, if leſs obliquely it is in a great meaſure tranſmitted [b]. Now it is not to be imagined that light at one degree of obliquity ſhould meet with pores enough in the air to tranſmit the greateſt part of it, and at another degree of obliquity ſhould meet with nothing but parts to reflect it wholly: eſpecially conſidering that in its paſſage out of air into glaſs, how oblique ſoever be its incidence, it finds pores enough in the glaſs to tranſmit a great part of it. If any man ſuppoſes that it is not reflected by the air, but by the outmoſt ſuperficial parts of the glaſs, there is ſtill the ſame difficulty: beſides that ſuch a ſuppoſition is unintelligible and will appear to be falſe by applying water behind ſome part of the glaſs inſtead of air. For ſo in a convenient obliquity of the rays ſuppoſe of 45 or 46 degrees, at which they are all reflected where air is adjacent to the glaſs, they ſhall be in a great meaſure tranſmitted where water is adjacent to it. Which argues that their reflection or tranſmiſſion depends on the conſtitution of air and water behind the glaſs and not on the ſtriking of the rays on the parts of the glaſs.

b Exp. 4. Art. 173.

Thirdly,

Thirdly, if the colours made by a prifm, placed at the entrance of a beam of light into a darkned room, be fucceffively caft on a fecond prifm placed at a great diftance from the former, in fuch manner that they are all alike incident upon it, (as they will be when tranfmitted through the holes in the two boards made ufe of in the 3d experiment,) the fecond prifm may be fo inclined to the incident rays, that thofe which are of a blue colour fhall be all reflected by it, and yet thofe of a red pretty copioufly tranfmitted. Now if reflection be caufed by the parts of air or glafs I would afk why at the fame obliquity of incidence, the blue fhould wholly impinge on thofe parts, fo as to be all reflected and yet the red find pores enough to be in a great meafure tranfmitted. Laftly were the rays of light reflected by impinging on the folid parts of bodies, their reflections from polifhed bodies could not be fo regular as they are. For in polifhing glafs with fand, putty or tripoli, it cannot be imagined that thofe fubftances can by grating and fretting the glafs bring all its leaft particles to an accurate polifh; fo that all their furfaces fhall be truly plane or truly fpherical, and look all the fame way, fo as together to compofe one even furface. This manner of polifhing with powders can do no more than bring the roughnefs of the glafs to a very fine grain, fo that the fcratches and frettings of the furface become too fmall to be vifible. And therefore if light were reflected by impinging upon the folid parts of the glafs, it would be fcattered as much by the moft polifhed glafs as by the roughest. So then it remains a problem how glafs polifhed by fretting fubftances can reflect light fo regularly as it does.

Fig. 280.

184. And this problem is fcarce otherwife to be folved than by faying that the reflection of a ray is effected not by a fingle point of the reflecting body, but by fome power of the body which is evenly diffufed all over its furface and by which it acts upon a ray without immediate contact. For that the parts of bodies do act upon light at a diftance, will appear by the following experiments.

But by an active power diffufed over its furface.

185. The fun fhining into my chamber through a hole a quarter of an inch broad, I placed at the diftance of two or three feet from the hole a fheet of paftboard, which was blacked all over on both fides, and in the middle of it had a hole about three quarters of an inch fquare for the light to pafs through. And behind the hole I faftened to the paftboard with pitch the blade of a fharp knife, to intercept fome part of the light which paffed through the hole. The planes of the paftboard and of the knife were parallel to one another and perpendicular to the rays. And when they were fo placed that none of the fun's light fell upon the paftboard, but all of it paffed through the hole to the knife, and there part of it fell upon the blade of the knife, and part of it paffed by its edge; I let this part of the light, which paffed by, fall on a white paper two or three feet beyond the knife, and there faw two ftreams of faint light fhoot out

XIII. Experiment. This power acts upon light at a diftance by attracting and repelling it. Newt. Opt. p. 300.

both

both ways from the beam of light into the ſhadow, like the tails of comets. But becauſe the ſun's direct light by its brightneſs upon the paper obſcured theſe faint ſtreams ſo that I could ſcarce ſee them, I made a little hole in the midſt of the paper for that light to paſs through, and fall upon a black cloth behind it, and then I ſaw the two ſtreams plainly. They were like one another and pretty nearly equal in length and breadth, and quantity of light. Their light at that end next the ſun's direct light was pretty ſtrong for the ſpace of about a quarter of an inch or half an inch, and in all its progreſs from that direct light decreaſed gradually till it became inſenſible. The whole length of either of theſe ſtreams meaſured upon the paper, at the diſtance of three feet from the knife, was about ſix or eight inches; ſo that it ſubtended an angle at the edge of the knife of 10 or 12, or at moſt 14 degrees.

XIV. Experiment.

I placed another knife by this, ſo that their edges might be parallel and look towards one another, and that the beam of light might fall upon both knives and ſome part of it paſs between their edges. And when the diſtance of their edges was about the 400th part of an inch, the ſtream parted in the middle and left a ſhadow between the two parts. This ſhadow was ſo black and dark that all the light which paſſed between the two knives ſeemed to be bent and to be turned aſide to the one hand and to the other. And as the knives ſtill approached one another the ſhadow grew broader, and the ſtreams ſhorter at their inward ends next the ſhadow, until upon the contact of the knives the whole light vaniſhed and left its place to the ſhadow. And hence I gather that the light which is leaſt bent, and goes to the inward ends of the ſtreams, paſſes by the edges of the knives at the greateſt diſtance, and this diſtance when the ſhadow begins to appear between the ſtreams is about the 800th part of an inch. And the light which paſſes by the edges of the knives at diſtances ſtill leſs and leſs is more and more bent, and goes to thoſe parts of the ſtreams which are farther and farther from the direct light. Becauſe when the knives approach one another till they touch, thoſe parts of the ſtreams vaniſh laſt which are fartheſt from the direct light.

Our author has made it appear from theſe and ſome other experiments, that bodies act upon light in ſome circumſtances by an attractive and in others by a repulſive power. For he found that the ſhadows of hairs, threds, pins, ſtraws, and ſuch like ſlender ſubſtances, placed in a ſlender beam of light let into a dark room, were conſiderably broader than they ought to be, if the rays of light paſſed on by theſe bodies in right lines. Particularly he found that the ſhadow of a hair of a man's head, at the diſtance of 10 feet from the hair, was 35 times broader than the hair it ſelf [a]. And of theſe attractive and repulſive powers he declares his ſentiments more fully in theſe words [b]. Since metals diſſolved in acids attract but a ſmall quantity of the acid, their attractive force can reach but to a ſmall diſtance

[a] Newt. Opt. p. 293.
[b] Ibid. p. 370.

distance from them. And as in algebra where affirmative quantities vanish and cease there negative ones begin, so in mechanicks where attraction ceases there a repulsive virtue ought to succeed. And that there is such a virtue seems to follow from the reflections and inflections of the rays of light. For the rays are repelled by bodies in both these cases without the immediate contact of the reflecting or inflecting body[a]. It appears also from the emission of light: the ray so soon as it is shaken off from a shining body, by the vibrating motion of the parts of the body, and gets beyond the reach of attraction, being driven away with exceeding great velocity. For that force which is sufficient to turn it back in reflection, may be sufficient to emit it. It seems also to follow from the production of air and vapours. The particles when they are shaken off from bodies by heat or fermentation, so soon as they are beyond the reach of the attraction of the body, receding from it and also from one another with great strength, and keeping at a distance so as sometimes to take up a million of times more space than they did before in the form of a dense body. Which vast contraction and expansion seems unintelligible by feigning the particles of air to be springy and ramous or rolled up like hoops or by any other means than by a repulsive power. From this power it seems to be that flies walk upon water without wetting their feet, and that the object-glasses of long telescopes lye upon one another without touching; and that two polished marbles which by immediate contact stick together, are difficultly brought so close together as to stick.

a Experiment 13, 14.

And is infinitely stronger than the power of gravity.

186. That this power which acts upon light is infinitely stronger than the power of gravity will appear by the following argument. Sir *Isaac Newton* has demonstrated that all bodies attract one another by the force of gravity, and that the attractive forces of two homogeneal spheres, upon particles of matter placed very near their surfaces, are to each other in proportion as the diameters of the spheres[1]. That is to say, if a refracting medium be spherical and of the same density as the earth, the earths force of attraction near its surface, will exceed the mediums force near its surface, as much as the diameter of the earth exceeds the diameter of the medium; or almost infinitely with respect to human conceptions. Yet we observe that a cannon-ball, just shot from the mouth of the cannon, is scarce sensibly deflected towards the earth by its attraction; and the least particle of the ball, if it was separate from the rest, would be no more deflected than the whole; because gravity makes bodies of all sorts and sizes descend with the same swiftness, by affecting them alike whether joined or separated. Therefore a particle of light which moves, I may say, infinitely quicker than a cannon-ball[2], would be infinitely

1 Princip. lib. 1. prop. 74. cor. 2 & lib. 3. prop. 8.
2 See velocity of light in the Index of this book.

leſs bent than the particle of the ball by the attraction of the whole earth, and ſtill infinitely leſs, than this laſt bending, by the attraction of the ſpherical medium, which was ſhewn to be infinitely weaker than that of the earth. But in fact we find it is very ſenſibly bent or refracted; and therefore it muſt be affected by ſome other power of the medium, which near its ſurface is infinitely ſtronger than the power of gravity.

And decreaſes much quicker.

187. It is difficult to determine the exact law of this refractive power, or the degrees of its force at given diſtances from the refracting ſurface. However ſince we find that the effects of gravity, which decreaſe as the ſquares of the diſtances from the center increaſe, are very ſenſible at great diſtances, we may conclude that the refractive power of a medium, which at its ſurface we find is infinitely ſtronger than gravity, and yet vaniſhes at a very ſmall diſtance from it[a], decreaſes much quicker or in a greater proportion than gravity does.

a Art. 185.

This one power both refracts and reflects light. Newt. Opt. p. 244.

188. It is reaſonable to conclude that bodies reflect and refract light by one and the ſame power variouſly exerciſed in various circumſtances; becauſe when light goes out of glaſs into air as obliquely as it can poſſibly do, if its incidence be made ſtill more oblique, it becomes totally reflected[b]: (for the power of the glaſs after it has reflected the light as obliquely as is poſſible, if the incidence be ſtill made more oblique becomes too ſtrong to let any of its rays go through and by conſequence cauſes total reflections:) And for this other reaſon, that thoſe ſurfaces of tranſparent bodies which have the greateſt refracting power, reflect the greateſt quantity of light, as will be ſhewn in the next chapter.

b Art. 173.

Its forces in different bodies are as their denſities nearly. Newt. Opt. p. 245.

189. From the different ratios of the ſines of incidence and refraction in a great many different bodies, our author has alſo collected that the forces of bodies to reflect and refract light are very nearly proportionable to their denſities, excepting that unctuous and ſulphureous bodies refract more than others of the ſame denſity. Whence, he ſays, it ſeems rational to attribute the refractive power of all bodies chiefly, if not wholly to the ſulphurous, oily particles with which they abound. For it is probable that all bodies abound more or leſs with ſulphurs. And as light congregated by a burning glaſs acts moſt upon ſulphureous bodies to turn them into fire and flame, ſo ſince all action is mutual, ſulphurs ought to act moſt upon light. For that the action between light and bodies is mutual, may appear from this conſideration; that the denſeſt bodies which refract and reflect light moſt ſtrongly grow hotteſt in the ſummer ſun, by the action of the refracted or reflected light. If bodies be conceived to have certain denſities exactly proportionable to their refractive powers, theſe may be called their refractive denſities.

Refractive denſities what.

This force acts in lines perpendicular to the refracting ſurface.

190. The direction of the refractive force of a medium, acting upon particles of light, is every where perpendicular to the refracting ſurface. For whether this force be a real attraction, or whether it be an impulſe upon

upon light, caused by the spring or elastick power of a subtil fluid which pervades the medium, and being gradually denser without than within it, may impel the light towards the medium by its greater elasticity without than within[a]; be this as you please, yet if the medium be uniform in all its parts, its immediate power upon the light it self, or upon the subtil fluid which acts upon it, will be equally strong in every point of a plane drawn parallel to the refracting surface; though its strength may be different in the next parallel plane, and still different in the next, and so on as far as that power is extended on each side of the surface of the medium. The extent of this power will therefore be terminated by two planes, parallel to one another and to the refracting surface; and the space between them may be called the space of activity, whether the power attracts or repels. This being premised, I say the force of the medium will act upon light, either in attracting or repelling it, in lines perpendicular to its surface. For let *p* be a particle of light acted upon by any uniform power in the line *de* parallel to the refracting surface *AB*, *pc* a line perpendicular to those parallels, cutting *de* in *c*; it is evident that the force of the power at *c* will move the particle *p* in the line *pc*; and taking any two points *d*, *e* at equal distances on each side of *c*, the powers at *d* and *e* being equal and acting at equal distances, *pd*, *pe*, equally inclined to *pc*, cannot move *p* in any direction but that of *pc*; and what has been said of the equal powers in the line *de* is applicable to the powers in every line drawn parallel to *AB*, that is to the whole power of the refracting medium.

a Newt. Opt. p. 323. &c.

Space of activity what.

Fig. 286, 287.

191. Now when a ray of light falls perpendicularly upon the space of activity, its particles will be accelerated or retarded in the same perpendicular direction, according as the power of the medium acts with or against the course of their motion; and when the particles are got through that space they will proceed with an uniform velocity. But if a ray *op* or *sr* falls obliquely upon the space of activity *klmn*, the force of the medium now acting sideways or obliquely upon the particles, will bend their course into a curve *pqr*, during their passage through that space. For as light has this property in common with all other bodies, of moving straight forwards, while its motion is not disturbed by an oblique force, so when it is disturbed, we may reasonably conclude, it will follow those other laws of motion, to which all other bodies are equally subject. The force of the medium acting sideways upon its oblique course, will therefore draw it perpetually out of one direction into another. But having passed through the space of activity, it will then proceed straight forward; for being attracted or impelled every way alike, or else not at all if it be in empty space, it will have the same freedom of motion in both cases: just as an animal surrounded with air, though violently pressed on every side, feels no constraint, but has an equal facility of moving in any di-

The manner of its operation in causing refractions and reflections. Fig. 288.

rection. Thus we see that the refraction of light is performed in the same manner as if a stone was thrown in the direction *op*, and its course was bent into a curve *pqr* by its gravity; or being thrown the contrary way in the direction *sr*, it was bent into the curve *rqp* in ascending: and supposing the attraction of the earth to reach no higher than the line *kl*

Fig. 289.

the stone would from thence proceed in a straight line *po*. Now the gravity of the stone may be so great, or the force of projection so weak, or the direction of the motion so oblique to the horizontal line *kl*, that it cannot ascend so high as this line. In this case the stone will descend from the highest point of its course by the same degrees of curvity with which it ascended; and if its gravity be supposed to cease in all places below the line *mn*, the stone will go on in the direction of the last particle of the curve produced. This is a parallel case to that of reflections from the farther surface of dense mediums, when the incident ray is so much inclined to that surface as to be pulled back into the same medium. Hitherto I have supposed the refracting medium to be contiguous to empty space; but the manner of reflection and refraction is the same as the common surface of any two mediums. For since the separate forces of the mediums act in the same lines, perpendicular to their common surface [a], and in contrary directions; the light will be affected with the difference of those forces in the same manner as before. And if the mediums have equal forces they will ballance each other, without causing any reflection or refraction at all. It has been observed already that the perpendicular breadth of the space of activity is exceeding small, and consequently in physical experiments the incurvation of the ray may still be considered as performed in a physical point.

a Art. 190

And in causing the different refrangibility of rays. Newt. Opt. p. 347.

192. According to this theory nothing more is requisite for producing all the variety of colours and degrees of refrangibility, than that the rays of light be bodies of different sizes; the least of which may make violet, the weakest and darkest of the colours, and be more easily diverted by refracting surfaces from its right course; and the rest, as they are bigger and bigger, may make the stronger and more lucid colours, blue, green, yellow and red; and be more and more difficultly diverted. For particles of different sizes, that fall upon the space of activity *klmn* in the line *op*, having different forces, may describe different curves, as *pa*, *pb*, *pc*, and consequently will emerge from that space in different angles.

Fig. 290.

And in causing the angles of reflection to be the same in all sorts of rays. Fig. 290.

193. Thus may heterogeneal particles diverge from one another by refraction, though not by reflection. For if the line of their incidence *op* be so oblique to the space of attraction *klmn*, that all the particles are pulled back into the same medium, they will return in parallel lines *rs*, *tu*, *xy*, &c. inclined to that space in the same angles as the line of incidence *op* is inclined to it. Just as several balls of different sizes, shot with different forces out of a cannon *op* in any fixt position, will describe dif-

ferent

ferent curves, as *pdr*, *pet*, *pfx*, &c. yet in returning to the ground they will all ſtrike upon it in equal angles at *r*, *t*, *x*, &c. every one being equal to the angle of elevation at *p*. Now ſince the ſpace of attraction is exceeding thin, the parallel lines *rs*, *tu*, &c. will be ſo cloſe together that the ſenſe cannot perceive a diſtinct ſenſation of the ſeparated particles, and conſequently the reflected and incident light will appear of the ſame colour. And when the incident light conſiſts of ſeveral rays though the particles in each ray may be a little ſeparated after reflection, and proceed in different lines, yet thoſe ſeveral lines will be mixt together, and conſequently the reflected light will appear white or of the ſame colour as the incident light.

194. By what I have quoted from Sir *Iſaac Newton* in the 185th article, his notion of the cauſe and manner of reflection from opake bodies, and from the firſt ſurface of tranſparent bodies, ſeems to be this that follows. Let the attractive power of the denſe medium *ABCD* end at the line *kl*, and there let the repulſive power begin, and let it end at the parallel line *hi*; and when a ray *op* falls from air upon the ſpace of repulſion *hikl*, it will be perpetually diverting from one direction to another by the oppoſition of the repulſive force, and ſo will deſcribe a curve *pqr*, till it emerges from that ſpace in the ſame angle at *r* with which it immerged at *p*, and then it will proceed in a right line *rs*. This will be the courſe of the ray if its progreſſive force be but weak, or the repulſive force be ſo ſtrong as to hinder it from entering the ſpace of attraction *klmn*. For if it enters this ſpace, inſtead of being reflected, it will be refracted into the denſe medium. And in reality ſome part of the incident light is always reflected and ſome refracted at all tranſparent ſurfaces; the cauſe of which our author has alſo conſidered[a]. And in cauſing reflections from opake bodies and from the firſt ſurface of tranſparent ones. Fig. 291.

Hence it ſeems to follow that the repulſive power of a denſe medium is leſs extended or elſe weaker than the attractive. For if the bending of a ray by the repulſive power, was not leſs than the contrary bending made by the attractive, the refraction into a denſe medium could not always be made towards the perpendicular, as it always is. We may alſo obſerve that a refracting ray, in its paſſage through the ſurface of a tranſparent medium, is bent backward and forward with a motion like that of an eel; and our author takes notice of the ſame ſort of motion in its paſſage by the edges and ſides of bodies. It follows alſo that the repulſive power does not extend to a ſenſible diſtance from the medium; for if it did, it would be diſcovered by a ſenſible incurvation of the ray throughout that extent; contrary to experience.

[a] Newt. Opt. p. 253.

195. Thus ſays our author will nature be very conformable to her ſelf in performing all the great motions of the heavenly bodies by the attraction of gravity, which intercedes thoſe bodies, and almoſt all the ſmall ones of their particles, by ſome other attractive and repelling powers, The diſtinguiſhing character of Sir *Iſaac Newton's* philoſophy.

which

which intercede the particles[a]. Since bodies at reſt, or put in motion cannot alter their ſtate of reſt, or the direction of their motion; it follows that bodies which move in curves, as the planets do, are continually drawn or impelled out of one direction into another by ſome power acting conſtantly upon them: juſt as bodies projected in the air are drawn into curves by the power of gravity. Without aſſuming any other principles, our author has ſhewn by the ſtricteſt reaſoning, what effects this power muſt produce among comets, planets and ſatellits once put into motion; as what would be the ſhapes and poſitions of their orbits; with what velocities they would move in different places; what proportions there would be between the times of their periods about a central body, with reſpect to their diſtances from it; what diſturbances they would cauſe in each others motions, and in the flux and reflux of the ſea, and the like. In a word he has exactly deſcribed all the particulars of a planetary ſyſtem. Then by comparing this ſyſtem with facts and obſervations, made upon the real motions, periods and diſtances of the planets and comets, he has ſhewn ſo exact an agreement between them, not in groſs but quantity for quantity, in a thouſand particulars, that whatever differences appeared at firſt, are always found to be leſs and leſs by more accurate obſervations. Now as ſo exact a conformity of reaſon and experience is the greateſt evidence we can poſſibly have, that this explication of the mechaniſm of the world is a true explication; and as it is the diſtinguiſhing character of his ſyſtem, which all the philoſophers that lived before him could never give of any of theirs, for want of ſufficient ſkill in geometry; ſo likewiſe he has given us the ſame evidence for the truth of his theory of light. For inſtance, having ſhewn by experiments that light and bodies affect one another at a diſtance, by ſome power that intercedes them; whatever be the law of this power, he has proved mathematically, by two different methods, that in all refractions the ſine of incidence muſt be to the ſine of refraction in a given ratio, and in reflections that the angles and ſines are equal[b]. I have made this compariſon more at large to rectify an opinion ſtill current, even amonſt men of learning; who obſerving that various ſyſtems of philoſophy in ſucceſſive ages have riſen and ſunk in the opinions of men, like modes and faſhions, and not conſidering in what reſpect our author's ſyſtem is plainly diſtinguiſhed from that of all others, are apt to think that his alſo in time will ſuffer the ſame fate. But it muſt be allowed, ſo long as there is a conſtancy in the nature of things, and a mutual agreement of reaſon and experience, that a ſyſtem founded entirely upon this alone and nothing elſe can never be changed, nor be leſs eternal than the world and truth it ſelf. I will conclude this book with our author's diſcoveries concerning the particular conſtitutions of bodies; which render ſome of them tranſparent, others opake and coloured.

[a] Newt. Opt. p. 372.

[b] Opt. p. 68. Princip. lib. I. prop. 94.

CHAP.

CHAP. VIII.

CONCERNING TRANSPARENCY, OPACITY AND COLOURS IN BODIES.

196. THOSE superfices of transparent bodies reflect the greatest quantity of light which have the greatest refractive power; that is which intercede mediums that differ most in their refractive densities[a]: and in the confines of equally refracting mediums there is no reflection. The analogy between refraction and reflection will apppear by considering, that when light passes obliquely out of one medium into another which refracts from the perpendicular, the greater is the difference of their refractive densities, the less obliquity of incidence is requisite to cause a total reflection[b]. Those superfices therefore which refract most, do soonest reflect all the light which is incident upon them, and so must be allowed most strongly reflective. But the truth of this proposition will farther appear by observing, that in the superfices interceding two transparent mediums (such as are air, water, oil, common glass, crystal, metalline glass, island glass, white transparent arsenick, diamonds, &c.) the reflection is stronger and weaker accordingly as the superfices hath a greater or less refractive power. For in the confine of air and sal-gem it is stronger than in the confine of air and water, and still stronger in the confine of air and common glass or crystal, and stronger in the confine of air and a diamond. If any of these and such like transparent solids, be immerged in water, its reflection becomes much weaker than before, and still weaker if they be immerged in the more strongly refracting liquors of well rectified oil of vitriol or spirit of turpentine. If water be distinguished into two parts by an imaginary surface, the reflection in the confine of these two parts is none at all; in the confine of water and ice it is very little; and in that of water and oil it is something greater; in that of water and sal-gem still greater, and in that of water and glass or crystal or other denser substances still greater, accordingly as those mediums differ more or less in their refractive powers. Hence in the confine of common glass and crystal there ought to be a weak reflection, and a stronger reflection in the confiine of common and metalline glass, though I have not yet tried this. But in the confine of two glasses of equal densities, as of two object-glasses of long telescopes pressed gently together, there is not any sensible reflection. For objects may be seen by rays obliquely transmitted through a round black spot where the glasses touch one another, but not through other places where the light is reflected at the interval between the glasses. And the same may be understood of the superficies interceding two crystals, or two liquors, in which no reflection is caused. So then the reason why uniform pellucid

Stronger and weaker reflections how caused.

Newt, Opt. p. 220.

a Art. 189.

b Art. 17.

cid mediums, ſuch as water, glaſs, or cryſtal, have no ſenſible reflection, but in their external ſuperficies, where they are adjacent to other mediums of a different denſity, is becauſe all their contiguous parts have one and the ſame degree of denſity: or this uniform denſity of their contiguous parts is a neceſſary condition of the traſparency of the whole.

Opacity cauſed by a multitude of internal reflections. Newt. Opt. p. 222.

197. The leaſt parts of almoſt all natural bodies are in ſome meaſure tranſparent: and the opacity of thoſe bodies ariſeth from the multitude of reflections cauſed in their internal parts. That this is ſo hath been obſerved by others, and will eaſily be granted by them that have been converſant with microſcopes. And it may alſo be tried by applying any ſubſtance to a hole, through which ſome light is immitted into a dark room. For how opake ſoever that ſubſtance may ſeem in the open air, it will by that means appear very manifeſtly tranſparent, if it be of a ſufficient thinneſs. Only white metalline bodies muſt be excepted, which by reaſon of their exceſſive denſity ſeem to reflect almoſt all the light incident on their firſt ſuperficies, unleſs by ſolution in menſtruums they be reduced into very ſmall particles, and then they become tranſparent.

The conſtitution of opake and coloured bodies what. Newt. Opt. p. 223.

698 Between the parts of opake and coloured bodies are many ſpaces, either empty or replenished with mediums of other denſities; as water between the tinging corpuſcles wherewith a liquor is impregnated; air between the aqueous globules that conſtitute clouds or miſts; and for the moſt part ſpaces void of both air and water, but yet perhaps not wholly void of all ſubſtance, between the parts of hard bodies. The truth of this is evident by the two preceeding articles. For by the latter article there are many reflections made by the internal parts of bodies, which by the former article would not happen if the parts of theſe bodies were continued, without any ſuch interſtices between them: becauſe reflections are cauſed only in ſuperficies which intercede mediums of a different denſity by article 196.

But farther, that this diſcontinuity of parts is the principle cauſe of the opacity of bodies, will appear by conſidering, that opake ſubſtances become tranſparent by filling their pores with any ſubſtance of equal or almoſt equal denſities with their parts. Thus paper dipt in water or oil, the *oculus mundi* ſtone ſteeped in water, linen cloth oiled or varniſhed, and many other ſubſtances ſoaked in ſuch liquors as will intimately pervade their little pores, or ſeparating parts, become by that means more tranſparent then otherwiſe. So on the contrary, the moſt tranſparent ſubſtances may by evacuating their pores, or ſeparating parts, be rendered ſufficiently opake; as ſalts or wet paper or the *oculus mundi* ſtone by being dried; horn by being ſcraped; glaſs by being reduced to powder or otherwiſe flawed; turpentine by being ſtirred about with water till they mix imperfectly; and water by being formed into many ſmall bubbles, either alone in the form of froth, or by ſhaking it together with oil

oil of turpentine or oil of olive or with ſome other convenient liquor, with which it will not perfectly incorporate.

199. The parts of bodies and their interſtices muſt not be leſs than of ſome definite bigneſs, to render them opake and coloured. For the opakeſt bodies, if their parts be ſubtily divided, (as metals by being diſſolved in acid menſtruums &c.) become perfectly tranſparent; and at the ſuperficies of the object-glaſſes, mentioned in the 196th article, where they were very near to one another though they did not abſolutely touch, there was no ſenſible reflection. And likewiſe if a bubble be blown with water firſt made tenacious by diſſolving a little ſoap in it, and be covered with a clear glaſs, to defend it from being agitated by the external air, and be ſuffered to reſt a while, till by the continual ſubſiding of the water it becomes very thin; at the top where it is thinneſt, there will grow a round, black, ſpot (like that between the object-glaſſes) which will continually dilate it ſelf more and more till the bubble breaks; now this ſpot appears black and tranſparent for want of a ſenſible reflection, whereas the ſides of the bubble which are thicker than the top appear coloured and opake by a ſtrong reflection.

The conſtitution of tranſparent bodies what. Newt. Opt. p. 225.

On theſe grounds I perceive it is that water, ſalt, glaſs, ſtones and ſuch like ſubſtances are tranſparent. For upon diverſe conſiderations they ſeem to be as full of pores or interſtices between their parts as other bodies are, but yet their parts and interſtices to be too ſmall to cauſe reflections in their common ſurfaces.

200. The black ſpots at the top of the water-bubble, and in the middle of the object-glaſſes compreſſed together, are always ſurrounded by a multitude of concentrick rings of all ſorts of colours: and as the colour in every ring is the ſame quite round its circumference, and different in different rings, ſo it is manifeſt (from the ſpherical figure both of the object-glaſſes and of the bubble of water, and from the uniform gravity of the particles of the water ſubſiding gradually on all ſides from the top to the bottom) that the thickneſſes, both of the plate of air between the glaſſes and of the water-bubble, are alſo the ſame in every part of the ſame ring, and different in different rings. Which ſhews that the particular colour of any ring depends upon a particular thickneſs of the plate of air or ſhell of water, where the incident light of the open air is reflected to the eye. Rings of colours do alſo appear by light tranſmitted through the water-bubble, and through the object-glaſſes, held between the eye and the light, but their colours are different from thoſe that appear in the ſame places, by reflected light.

Rings of colours in water-bubbles &c. explained Newt. Opt. p. 168.

Theſe are the general appearances of the rings in the open air; but when homogeneal light is caſt by a priſm upon the object-glaſſes in a dark room, the colour of all the rings, ſeen by light reflected from the glaſſes, is the ſame as of the light thrown upon them; and in the inter-

vals between the coloured rings other dark rings appear like the ſpot in the middle; through all which the incident light is tranſmitted, and forms other intermediate rings of the ſame colour upon a white paper held behind the glaſſes, as repreſented in the figure. It is alſo to be obſerved that the diameters, breadths and intervals of rings of homogeneal lights, of different colours, are all different; thoſe made by homogeneal red being the largeſt, and by homogeneal violet the leaſt; and thoſe of the intermediate priſmatick colours are of intermediate ſizes. And from hence the origin of the different coloured rings in the open air is manifeſt; namely, that the air between the glaſſes, according to its various thickneſs, is diſpoſed in ſome places to reflect and in others to tranſmit the light of any one colour; and in the ſame place to reflect that of one colour where it tranſmits that of another. The appearances are the ſame when water is between the object-glaſſes, only the rings are ſmaller.

Fig. 294.

The conſtitution of coloured bodies farther explained. Newt. Opt. p. 226.

201. The tranſparent parts of bodies according to their ſeveral ſizes reflect rays of one colour and tranſmit thoſe of another; on the ſame grounds that thin plates or bubbles do reflect or tranſmit thoſe rays. And this I take to be the ground of all their colours. For if a thinned or plated body, which being of an even thickneſs appears all over of one uniform colour, ſhould be ſlit into threads or broken into fragments, I ſee no reaſon why every thread or fragment ſhould not keep its colour; and by conſequence why a heap of thoſe threads or fragments ſhould not conſtitute a maſs or powder of the ſame colour which the plate exhibited before it was broken. And the parts of all natural bodies being like ſo many fragments of a plate muſt on the ſame grounds exhibit the ſame colour.

Now that they do ſo will appear by the affinity of their properties. The fineſt coloured feathers of ſome birds, and particularly thoſe of peacocks tails, do in the very ſame part of the feather appear of ſeveral colours in ſeveral poſitions of the eye; after the ſame manner as the rings do in the water-bubble and between the object-glaſſes; and therefore their colours ariſe from the thinneſs of the tranſparent parts of the feathers; that is from the ſlenderneſs of the very fine hairs, or *capillamenta*, which grow out of the ſides of the groſſer lateral branches or fibres of thoſe feathers. And to the ſame purpoſe it is that the webs of ſome ſpiders, by being ſpun very fine, have appeared coloured, as ſome have obſerved; and that the coloured fibres of ſome ſilks by varying the poſition of the eye do vary their colours. Alſo the colours of ſilks, cloths and other ſubſtances, which water or oil can intimately penetrate, become more faint and obſcure by being immerged in thoſe liquors; and recover their vigour again by being dried; much after the ſame manner as the colours of thin plates of muſcovy glaſs become more faint and languid by wetting them, and alſo thoſe of the rings by letting water creep in between the object-glaſſes. Leaf gold and ſome ſorts of painted glaſs, and the infuſion of

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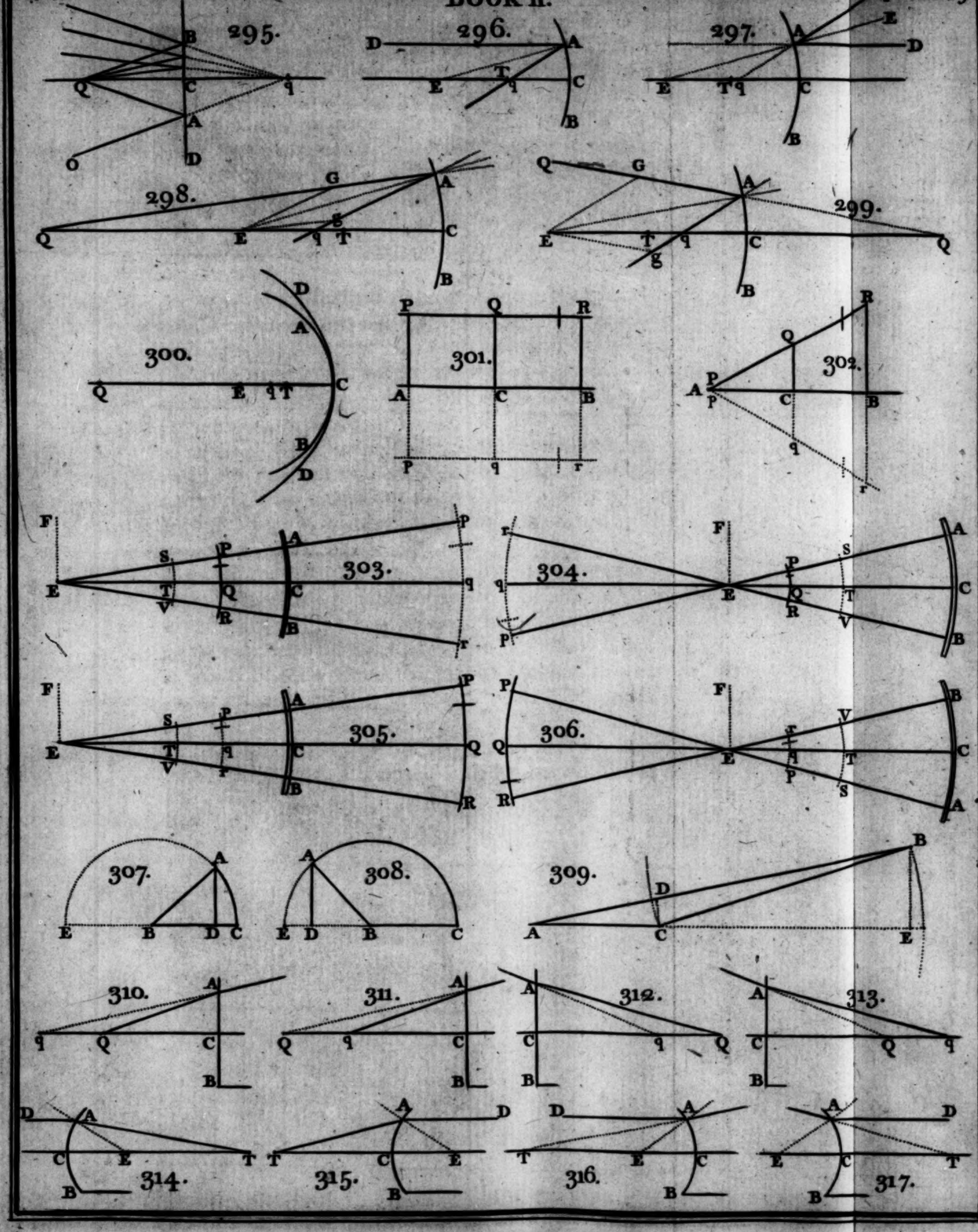
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lignum nephriticum, reflect one colour and transmit another, as the rings in the water-bubbles and between the object-glasses do in the very same places. and some of those coloured powders which painters use, may have their colours a little changed by being very elaborately and finely ground. Where I see not what can be justly pretended for those changes, besides the breaking of their parts into less parts by that contrition, after the same manner that the colour of a thin plate is changed by varying its thickness. For which reason it is that the coloured flowers of plants and vegetables by being bruised usually become more transparent than before, or at least in some degree or other change their colours. Nor is it much less to my purpose, that by mixing divers liquors very odd and remarkable productions and changes of colours may be effected [1]; of which no cause can be more obvious and rational than that the saline corpuscles of one liquor, do variously act upon or unite with the tinging corpuscles of another; so as to make them swell, or shrink, (whereby not only their bulk but their density may also be changed;) or to divide them into smaller corpuscles, (whereby a coloured liquor may become transparent [a];) a Art. 199.
or to make many of them associate into one cluster, whereby two transparent liquors may compose a coloured one [b]. For we see how apt those b Art. 198.
saline menstruums are to penetrate and dissolve substances to which they are applied, and some of them to precipitate what others dissolve. In like manner, if we consider the various phænomena of the atmosphere, we may observe, that when vapours are first raised, they hinder not the transparency of the air, being divided into parts too small to cause any reflection in their superficies [c]. But when in order to compose drops of rain c Art. 199.
they begin to coalesce and constitute globules of all intermediate sizes, these globules when they become of a convenient size to reflect some sorts of rays and transmit others, may constitute clouds of various colours according to their sizes. And I see not what can be rationally conceived, in so transparent a substance as water, for the production of these colours besides the various sizes of its fluid and globular parcels.

1 See *Boyle* upon colours, experim. 20. p. 245.

A COMPLEAT SYSTEM OF OPTICKS.

BOOK II.

A *Mathematical Treatise.*

CHAPTER I.

To find the focus of rays reflected from any given surface.

PROPOSITION I.

Fig. 295. 202. LET ACB *be a reflecting plane, and* Q *the focus of the incident rays, and* QC *a perpendicular to that plane; and if this perpendicular be produced to* q, *so that* qC *be equal to* QC, *the point* q *shall be the focus of the reflected rays.*

For let QA be any incident ray; draw qA and produce it towards O, and CA towards D. Then because Cq is made equal to CQ, the triangles CAq, CAQ will be equal[a]. And consequently the angle DAO, which is equal to the opposite angle CAq, is also equal to the angle CAQ. Therefore AO is the reflected ray[b]. *Q. E. D.*

a 8th Ax. Eucl.
b Art. 9.

203. *Corol.* Hence all the rays that flow towards q, will flow to Q after reflection[c].

c Art. 11.

LEMMA.

204. *Quantities and their proportions, which so approach to a state of equality as to become equal at last, may be taken for equal in a state immediately preceding the last; and also in a state somewhat remote from the last without sensible error in physical subjects: and the same may be said of figures which continually approach to a state of similitude; especially if these errors, when computed, are found inconsiderable.*

The meaning of the lemma will appear very plain when applied to the following propositions.

PROPOSITION II.

Fig. 296, 297. 205. *When parallel rays as* DA, EC *fall almost perpendicularly upon a spherical surface* ACB, *the focus,* T, *of the reflected rays will bisect that semidiameter* EC, *which is parallel to the incident rays.*

For

For drawing EA, it will be perpendicular to the ſpherical ſurface at A, and ſince EC is in the ſame plane as the angle of incidence DAE, the reflected ray Aq (produced backwards in fig. 297.) will meet EC ſomewhere in q[a]; ſo that the angle of reflection EAq may be equal[b] to the angle of incidence EAD or to the alternate angle AEq. The two ſides Aq, Eq of the triangle AqE are therefore equal to each other[c]; and conſequently each of them greater than half the third ſide EA, or than ET by conſtruction. But as the point of incidence at A approaches towards C the lines Eq, ET continually approach towards equality, and become equal when the triangle AEq is vaniſhing: and ſo the focus of rays falling almoſt perpendicularly on the ſurface, or the neareſt to the point C, is to be reckoned at T[d]. *Q. E. D.*

a Art. 7. b Art. 8. c Euc. I. 6. d Art. 204.

206. *Corol.* Hence if T be the focus of the incident rays, the reflected ones will go parallel to the line TE[e]. e Art. 11.

PROPOSITION III.

207. *Let* ACB *be a reflecting ſurface of any ſphere whoſe center is* E. *Biſect any radius thereof, ſuppoſe* EC, *in* T; *and if in this radius, on the ſame ſide of the point* T, *you take the points* Q *and* q, *ſo that* TQ, TE *and* Tq *be continual proportionals; and the point* Q *be the focus of the incident rays, the point* q *ſhall be the focus of the reflected ones.* Fig. 298, 299.

Let QA and Aq be an incident and reflected ray (produced) making equal angles with the perpendicular AE; and the reflected ray Aq (produced) will cut QE (produced) ſomewhere in q, as being in the plane of incidence EAQ[f]. Draw EG parallel to Aq, and let it meet AQ in G; and alſo Eg parallel to AQ, meeting Aq in g: then becauſe the angles EAG, EAg are equal[g], it follows that the triangles EAG, EAg are equiangular at their common baſe AE[h], and therefore equicrural[i]; and alſo equal to each other; and conſequently each ſide of the equilateral figure $AGEg$, in its vaniſhing ſtate when A comes to C, will be equal to half its diagonal AE[k], or by conſtruction to ET. Now becauſe the triangles GQE, gEq are equiangular[l], it will be as GQ to GE, ſo gE to gq[m]; that is, when the point A is coinciding with C, and conſequently the points G, g with T, as TQ to TE ſo TE to Tq[n]. *Q. E. D.*

f Art. 7. g Art. 8. h Euc. I. 29. i Euc. I. 6. k Art. 204. l Euc. I. 29. m Euc. VI. 4. n Art. 204.

208. *Corol.* 1. If q be the focus of incident rays, Q will be the focus of the reflected ones[o]. o Art. 11.

209. *Corol.* 2. The rays that belong to Q may be reckoned parallel when the diſtance TQ is infinite, and then by this propoſition its reciprocal Tq becomes nothing; which is the ſecond propoſition.

210. *Corol.* 3. Hence alſo we may deduce the firſt propoſition; for ſuppoſing Q the focus of incident rays upon the convex ſurface AB; ſince TQ, TC, Tq are continual proportionals, it is well known that their differences CQ, Cq muſt become equal when the lines themſelves are infinite- Fig. 299.

ly;

ly great; that is when the surface becomes a plane by removing its center to an infinite distance.

The figures serve for the cases of a convex surface supposing the incident rays to go backwards in the same lines produced through the surface.

211. By the demonstrations of the two last propositions, it appears that the focus of reflected rays there determined, is nothing else in strictness of geometry, but the intersection of the axis of the surface, that is of the ray passing through its center, and of the nearest rays to it: and also that other rays intersect the axis in different points farther and farther from that focus, as they fall farther and farther from the vertex of the surface. So that a spherical surface cannot possibly reflect all the incident rays to a single point. Neverthelefs when these aberrations of the remoter rays from the geometrical focus shall be considered, it will appear hereafter, that the density of their intersections, near that focus, is immensely greater than their density at any considerable distance from it. So that in physical things, the focus of all the rays, that fall almost perpendicular upon a spherical surface, may be considered as a physical point. And the same is to be understood of the focus of refracted rays, as will appear by the like sort of demonstrations.

212. Hence it appears that the focus of rays reflected from any curved surface whatever, must be reckoned the same as if they were reflected from a spherical surface of an equal curvity to that surface about the points of incidence. As if *CD* be any curve whatever, *C* the point of incidence, *CEQ* perpendicular to the curve, or to its tangent at *C*, *CE* the radius of a circle *ACB* of the same degree of curvity at *C*; the rays coming parallel to *CE*, will be reflected to the same focus *T* from either of the surfaces; and also the rays that flow from any point *Q*, will be reflected by either surface to the same focus *q*. Because we consider the focus of those rays only, that fall upon the common points of both curves about *C*, all the rest being dispersed much thinner into other places.

Fig. 300.

213. In all these propositions when the focuses *Q*, *q* lye on the same side of the reflecting surface, if the incident rays flow from *Q* the reflected ones will flow towards *q*; and if the incident rays flow towards *Q*, the reflected ones will flow from *q*; and the contrary happens when *Q* and *q* are on contrary sides of the surface. Because the incident and reflected rays go contrary ways.

CHAPTER

CHAPTER II.

To determine the place, magnitude and situation of images formed by reflected rays.

PROPOSITION I.

214. IMAGES *formed by reflections from a plane surface are similar and equal to the objects; and their parts have the same situation with respect to the backside of the plane as the parts of the object have with respect to its foreside.*

From any number of points P, Q, R of an object in any situation, draw the perpendiculars PA, QC, RB to the plane ACB, and produce them through it to the points p, q, r, each as far behind the plane as P, Q, R are before it. The points p, q, r being the respective focuses of the rays that belonged to P, Q, R [a], and being evidently in the same order, together with infinite others, will constitute an image of the object, equal to it in the whole and in every corresponding part, and alike situated: as will appear by conceiving the surface of the object, and of its image, divided into corresponding lines, such as PQR, pqr, by planes such as $PprR$ perpendicular to the reflecting plane; and by folding up or doubling each plane in its line of intersection, AB, with the reflecting plane. For the parts of each plane on each side of AB will exactly cover each other, as appears by the construction. *Q. E. D.* Fig. 301, 302.

a Art. 202.

PROPOSITION II.

215. *If an arch of a circle* PQR, *concentrick to a concave or convex spherical surface* AB, *be considered as an object, its image* pqr *will also be a similar concentrick arch, whose length will be to the length of the object, in the ratio of their distances from the common center* E; *and its situation will be erect or inverted, according as it is on the same or the opposite side of the center to the object.* Fig. 303 to 306.

For as the focus Q was found by making TQ, TE, Tq continual proportionals in the line QE drawn through the center [b]; so the focus p, of rays that belong to any other point P, is found by drawing PEA, and bisecting EA in S, and by making SP, SE, Sp continual proportionals. The two first terms of one proportion are severally equal to the two first of the other; and consequently the third terms Tq, Sp are equal; and thence Ep and Eq are equal. The same being true of every point of the circular object PQR, shews that its image pqr is a concentrick arch, similar to it, both being terminated by the same lines EPp, ERr; and consequently their lengths are in the same ratio as their semidiameters

b Art. 207.

EQ,

EQ, *Eq*. Laſtly according as the correſponding extremities *P* and *p*, of the object and image, are on the ſame or oppoſite ſides of the center *E*, they are alſo on the ſame or oppoſite ſides of their middle points *Q*, *q*, that is the image is accordingly erect or inverted. *Q. E. D.*

216. *Corol.* The ſmaller the circular object is with reſpect to its radius or diſtance from the center, the nearer it approaches in ſhape to a ſtraight line, and ſo does its ſimilar image. Conſequently a ſmall ſtraight object, placed at a good diſtance from the center of the glaſs, may be reckoned to have a ſtraight image very nearly: though in ſtrictneſs of geometry it is an arch of a conick ſection, whoſe determination may be ſeen in the remarks.

217. The images of all ſorts of objects may be determined, by finding the images of their out-lines, by the foregoing propoſitions. For inſtance, if the plane of the figures *PER*, *pEr* be turned round their common diameter *QEq*, the circular ſurface generated by *pqr* will be the image of the circular object generated by *PQR*: and if the ſame figures *PER*, *pEr* be moved a little about an axis *EF*, ſituated in their own plane, and perpendicular to the diameter *QEq*, the curvilinear figure generated by this motion of *pqr*, will be the image of a ſimilar figure generated by *PQR*. Becauſe the reflecting arch *ACB* generates the reflecting ſpherical ſurface at the ſame time.

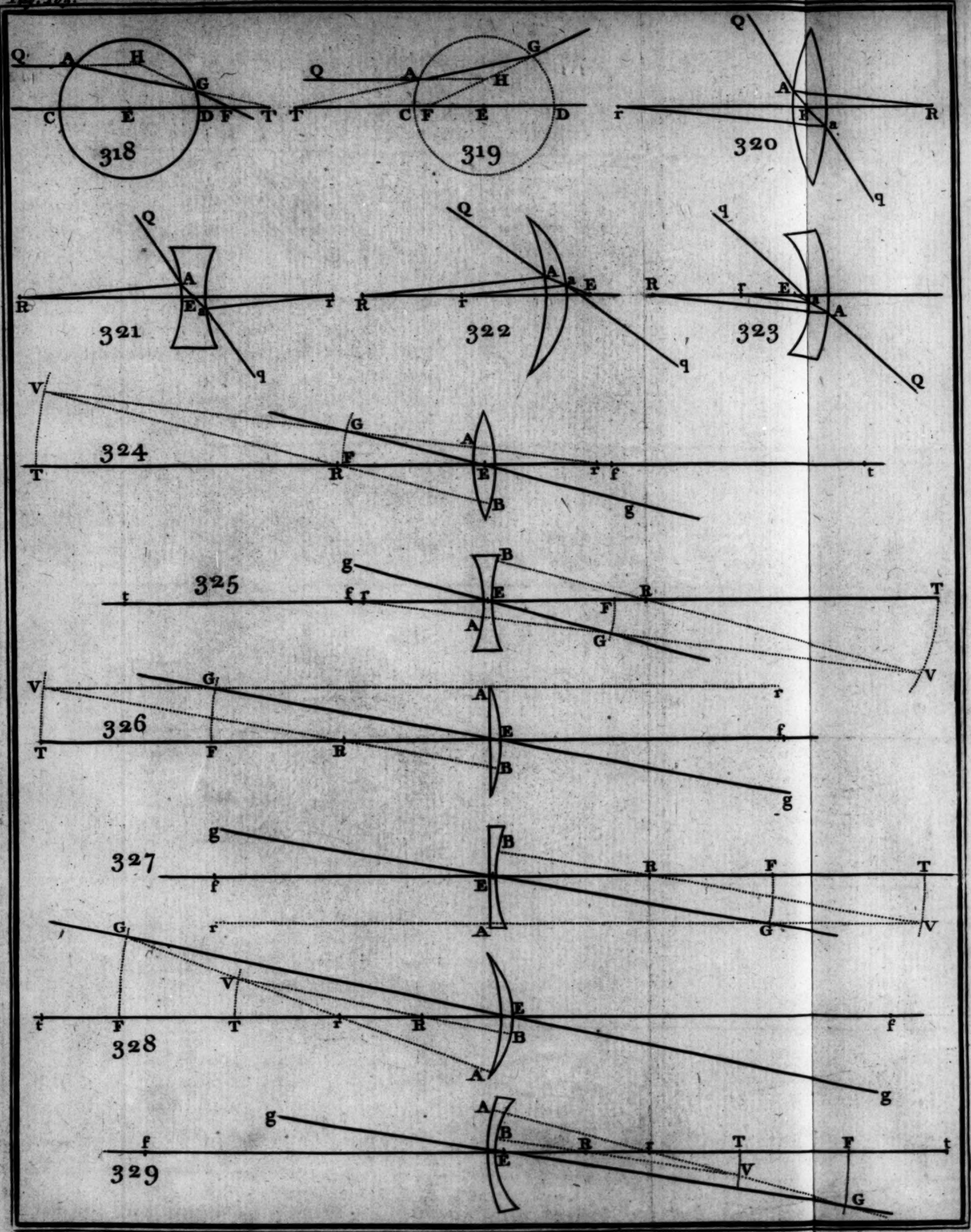
218. But if the whole figure *PERrp* be moved parallel to it ſelf in a direction *EF*, now perpendicular to its own plane, ſo that the arch *ACB* may generate a portion of a cylindrical ſurface, the figure deſcribed by this motion of *pqr*, will ſtill be the image of that deſcribed by *PQR*; but will not be ſimilar to it, except when they are placed at equal diſtances on each ſide the center *E*, and conſequently are equal to each other: and their diſſimilitude will be ſo much the greater as the diſproportion between *Eq* and *EQ* or between their lengths *pr*, *PR*, is greater; their breadths, deſcribed by the motion aforeſaid, being always equal to each other.

CHAPTER III.

To find the focus of rays falling almoſt perpendicularly upon any refracting ſurface, ſphere or lens.

DEFINITION.

219. THE ſine of an angle *ABC*, or of an arch *AC* that meaſures that angle, is a line *AD* drawn from the extremity of one of
Fig. 307, 308. the ſemidiameters, *AB*, *BC*, perpendicular to the other, produced if the angle be obtuſe. And therefore an angle *ABC* and its complement *ABE*, to two right angles, have each the ſame ſine *AD*; and when the ſines of ſeveral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ſeveral angles are compared together, they are always underſtood to belong to the ſame or to equal circles.

220. The ſines of very ſmall angles, and of their complements, become at laſt inſenſibly different from the arches that meaſure them; and conſequently are proportionable to the angles themſelves.

LEMMA.

221. *The ſines of the angles of any triangle are proportionable to the oppoſite ſides: as in the triangle* ABC, *the ſine of the angle* ABC *is to the ſine of the angle* BCA, *as* CA *to* AB. Fig. 309.

For the perpendiculars *CD*, *BE* upon thoſe ſides *AB*, *AC* produced, are the ſines of the angles *ABC*, *BCA* or *BCE* with reſpect to circles whoſe radius is *BC*[a]. And ſince the triangles *CAD*, *BAE* are equiangular[b], we have *CD* to *BE* as *CA* to *AB*. *Q.E.D.* a Art. 219. b Euc. I. 32. cor. 2.

222. *Corol.* Small angles, as *BAC*, *BCE*, ſubtended by the ſame perpendicular *BE*, are reciprocally as their legs *BA*, *BC* or *EA*, *EC*. For the angle *BAC* is to *BCE*, when very ſmall, as the ſine of *BAC* to the ſine of *BCE*[c], or as *BC* to *BA*[d] or as *EC* to *EA*[e]. c Art. 220. d Art. 221. e Art. 204.

PROPOSITION I.

223. *Let* ACB *be a refracting plane, and* Q *the focus of the incident rays, and* QC *a perpendicular to that plane; and if* qC *be taken in this perpendicular, on the ſame ſide of the plane as* QC, *and in proportion to* QC, *as the ſine of incidence to the ſine of refraction, the point* q *ſhall be the focus of the refracted rays.* Fig. 310 to 313.

For let the lines *QA* and *Aq*, produced as in the figures, repreſent an incident and a refracted ray, cutting *QC* in any point *q* whatever; and ſince a perpendicular to the plane at *A* is parallel to *QC*, the angle *AQC* will be equal to the angle of incidence, and *AqC* to the angle of refraction. Therefore ſince equal angles have equal ſines, the ſine of incidence is to the ſine of refraction (as the ſine of the angle *AQC*, to the ſine of *AqC*, or as *Aq* to *AQ*[f], that is when the ray *QA* is almoſt perpendicular to the plane *AB*) as *Cq* to *CQ*[g]. *Q.E.D.* f Art. 221. g Art. 204.

PROPOSITION II.

224. *Let* ACB *repreſent a refracting ſpherical ſurface whoſe center is* E, *and let the incident rays as* DA *come parallel to any ſemidiameter* CE, *in which produced forward or backward, according as the denſer medium is convex or concave, take* CT *to* CE *as the ſine of incidence to the difference of the ſines, and* T *will be the focus of the refracted rays.* Fig. 314. to 317.

For let the refracted ray *AT* (produced) cut the ſemidiameter *CE* produced, in any point *T* whatever; and ſince the ſemidiameter *EA* is perpendicular to the refracting ſurface at *A*, the angle of incidence will be equal to the angle *AEC*, and the angle of refraction, or its complement

to two right ones, will be *EAT*; consequently the sine of incidence is to the sine of refraction, (as the sine of the angle *AEC*, to the sine of *EAT*, or as *AT* to *TE*[a], that is, when *A* comes nearest to *C*, and so the incident rays are almost perpendicular to the surface,) as *CT* to *TE*[b]; and disjointly[c] the sine of incidence is to the difference of the sines as *CT* to *CE*. *Q.E.D.*

a Art. 221.
b Art. 204.
c Euc. V. 17.

225. *Corol.* 1. *CT* is to *TE* as the sine of incidence to the sine of refraction.

226. *Corol.* 2. If this point *T* be the focus of incident rays, the refracted rays will go parallel to *TE*[d].

d Art. 11.

PROPOSITION III.

Fig. 318, 319.

227. *When parallel rays fall upon a sphere, either denser or rarer than the ambient medium, in the diameter* CD *produced, which is parallel to the incident rays, as* QA, *let* T *be their focus after their first refraction at* AC; *and the point* F *which bisects* TD *shall be their focus after their second refraction at* DG.

For let the incident and emergent rays, *QA*, *FG* produced, meet in *H*, and since the refractions at *A* and *G* are equal, as appears by supposing a ray to go both ways along the chord *AG*, the triangle *AHG* is equiangular at its base *AG*, and therefore equicrural[e]; and so is the similar triangle *GFT*, the lines *AH*, *FT* being parallel. Therefore when *A* approaches toward *C* till *G* is coinciding with *D* and the triangle *GFT* is vanishing, the leg *GF* will become equal to half the base *GT*; that is *DF* will become equal to half *DT*[f]. *Q.E.D.*

e Euc. I. 6.
f Art. 204.

LEMMA.

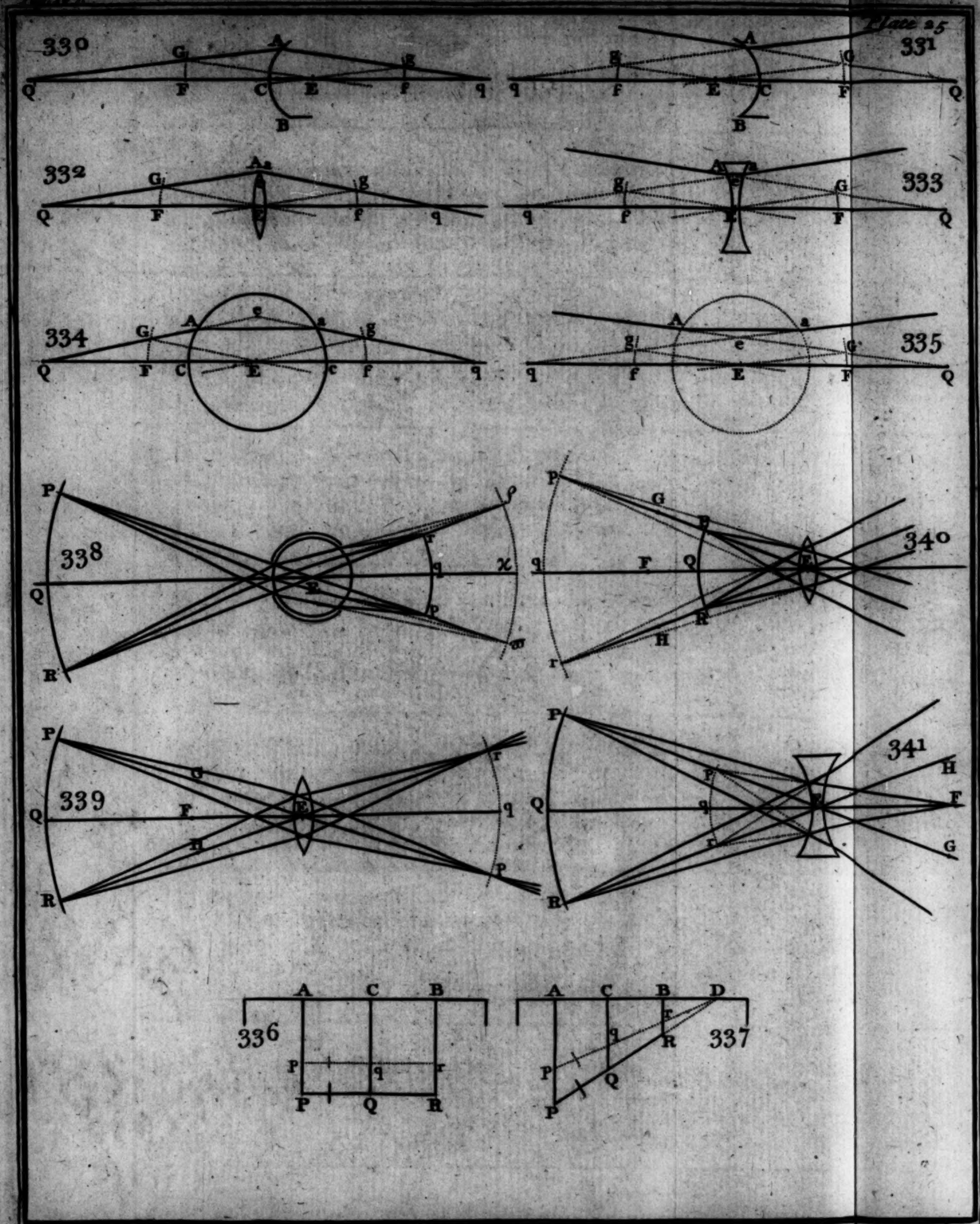
Fig. 320 to 323.

228. *There is a certain point* E *within every double convex or double concave lens, through which every ray that passes, will have its incident and emergent parts* QA, aq *parallel to each other: but in a plano-convex or plano-concave lens that point* E *is removed to the vertex of the concave or convex surface; and in a meniscus and in that other concavo-convex lens, it is removed a little way out of them, and lyes next to the surface which has the greatest curvity.*

For let *REr* be the axis of the lens joining the centers *R*, *r* of its surfaces *A*, *a*. Draw any two of their semidiameters *RA*, *ra* parallel to each other, and join the points *A*, *a*, and the line *Aa* will cut the axis in the point *E* above described. For the triangles *REA*, *rEa* being equiangular, *RE* will be to *Er* in the given ratio of the semidiameters *RA*, *ra*; and consequently the point *E* is invariable in the same lens. Now supposing a ray to pass both ways along the line *Aa*, it being equally inclined to the perpendiculars to the surfaces, will be equally bent and contrary ways in going out of the lens; so that its emergent parts *AQ*, *aq* will be parallel. Now any of these lenses will become plano-convex or plano-con-

cave,

Plate 25



cave, by conceiving one of the semidiameters RA, ra to become infinite and consequently to become parallel to the axis of the lens, and then the other semidiameter will coincide with the axis; and so the points A, E or a, E will coincide. *Q.E.D.*

229. *Corol.* Hence when a pencil of rays falls almost perpendicularly upon any lens, whose thickness is inconsiderable, the course of the ray which passes through E, above described, may be taken for a straight line passing through the center of the lens, without sensible error in sensible things. For it is manifest from the length of Aa and from the quantity of the refractions at its extremities, that the perpendicular distance of AQ, aq when produced, will be diminished both as the thickness of the lens and the obliquity of the ray is diminished.

PROPOSITION IV.

230. *To find the focus of parallel rays falling almost perpendicularly upon any given lens.*

Let E be the center of the lens, R and r the centers of its surfaces, Rr its axis, gEG a line parallel to the incident rays upon the surface B, whose center is R. Parallel to gE draw a semidiameter BR, in which produced let V be the focus of the rays after their first refraction at the surface B, and joining Vr let it cut gE produced in G, and G will be the focus of the rays that emerge from the lens. Fig. 324 to 329.

For since V is also the focus of the rays incident upon the second surface A, the emergent rays must have their focus in some point of that ray which passes straight through this surface; that is in the line Vr, drawn through its center r: and since the whole course of another ray is reckoned a straight line gEG[a], its intersection G with Vr determines the focus of them all. *Q.E.D.* a Art. 229.

231. *Corol.* 1. When the incident rays are parallel to the axis rR, the focal distance EF is equal to EG. For let the incident rays that were parallel to gE be gradually more inclined to the axis till they become parallel to it; and their first and second focuses V and G will describe circular arches VT and GF whose centers are R and E. For the line RV is invariable; being in proportion to RB in a given ratio of the lesser of the sines of incidence and refraction to their difference[b]; consequently the line EG is also invariable, being in proportion to the given line RV in the given ratio of rE to rR, because the triangles EGr, RVr are equiangular. b Art. 224.

232. *Corol.* 2. The last proportion gives the following rule for finding the focal distance of any thin lens. As Rr, the interval between the centers of the surfaces, is to rE, the semidiameter of the second surface, so is RV or RT, the continuation of the first semidiameter to the first focus, to EG or EF, the focal distance of the lens. Which according as the lens is

is thicker or thinner in the middle than at its edges, must lye on the same side as the emergent rays or the opposite side.

233. *Corol.* 3. Hence when rays fall parallel on both sides of any lens the focal distances EF, Ef are equal. For let rt be the continuation of the semidiameter Er to the first focus t of rays falling parallel upon the surface A; and the same rule that gave rR to rE as RT to EF, gives also rR to RE as rt to Ef. Whence Ef and EF are equal, because the rectangles under rE, RT and also under RE, rt are equal. For rE is to
a Art. 224. rt and also RE to RT in the same given ratio [a].

234. *Corol.* 4. Hence in particular in a double-convex or double-concave lens made of glass, it is as the sum of their semidiameters (or in a meniscus as their difference) to either of them, so is double the other, to the focal distance of the glass. For the continuations RT, rt are severally double their semidiameters: because in glass ET is to TR and also Et to
b Art. 225. 13. tr as 3 to 2 [b].

235. *Corol.* 5. Hence if the semidiameters of the surfaces of the glass be equal, its focal distance is equal to one of them; and is equal to the focal distance of a plano-convex or plano-concave glass whose semidiameter is as short again. For considering the plane surface as having an infinite semidiameter, the first ratio of the last mentioned proportion may be reckoned a ratio of equality.

Proposition V.

Fig. 330 to 335. 236. *The focus of incident rays upon a single surface, sphere or lens being given, it is required to find the focus of the emergent rays.*

Let any point Q be the focus of incident rays upon a spherical surface, lens, or sphere, whose center is E; and let other rays come parallel to the line QEq the contrary way to the given rays, and after refraction let them belong to a focus F; then taking Ef equal to EF, say as QF to FE so Ef to fq, and placing fq the contrary way from f to that of FQ from F, the point q will be the focus of the refracted rays, without sensible error; provided the point Q be not so remote from the axis, nor the surfaces so broad as to cause any of the rays to fall too obliquely upon them.

For with the center E and semidiameter EF or Ef describe two arches FG, fg cutting any ray $QAaq$ in G and g, and draw EG and Eg. Then supposing G to be a focus of incident rays, (as GA,) the emergent rays
c Art. 226. 231. 233. (as agq) will be parallel to GE [c]; and on the other hand supposing g another focus of incident rays (as ga,) the emergent rays (as AGQ,) will be parallel to gE. Therefore the triangles QGE, Egq are equiangular, and consequently QG is to GE as Eg to gq; that is, when the ray $QAaq$ is
d Art. 204. the nearest to QEq, QF is to FE as Ef to fg [d]. Now when Q accedes to F and coincides with it, the emergent rays become parallel, that is q recedes

recedes to an infinite distance; and consequently when Q passes to the other side of F, the focus q will also pass through an infinite space from one side of f to the other side of it. $Q.E.D.$

237. *Corol.* 1. In refractions at a spherical surface AC, the focus q may also be found by this rule, as QF to FC so Cf to fq; because FC and ef and also FE and Cf are equal [a]. a Art. 225.

238. *Corol.* 2. It may also be found by this rule, as QF to QE so QC to Qq; placing Qq so that all the four distances from Q may lye on one side of it, or else two on each. For the triangles QGE, QAq being equiangular we have QG to QE as QA to Qq.

239. *Corol.* 3. In a sphere or lens the focus q may be found by this rule, as QF to QE so QE to Qq, to be placed the same way from Q as QF lyes from Q. For let the incident and emergent ray QA, qa be produced till they meet in e; and the triangles QGE, Qeq being equiangular, we have QG to QE as Qe to Qq; and when the angles of these triangles are vanishing, the point e will coincide with E; because in the sphere the triangle Aea is equiangular at the base Aa, and consequently Ae and ae will at last become semidiameters of the sphere. In a lens the thickness Aa is inconsiderable.

240. *Corol.* 4. In all cases the distance fq varies reciprocally as FQ does; and they lye contrary ways from f and F; because the rectangle or the square under EF and Ef, the middle terms in the foregoing proportions, is invariable.

241. *Corol.* 5. Convex lenses of different shapes that have equal focal distances, when put into each others places, have equal powers upon any pencil of rays to refract them to the same focus. Because the rules abovementioned depend only upon the focal distance of the lens, and not upon the proportion of the semidiameters of its surfaces.

242. *Corol.* 6. The rule that was given for a sphere of an uniform density, will serve also for finding the focus of a pencil of rays refracted through any number of concentrick surfaces, which intercede uniform mediums of any different densities. For when rays come parallel to any line drawn through the common center of these mediums, and are refracted through them all, the distance of their focus from that center is invariable, as in an uniform sphere.

243. *Corol.* 7. When the focuses Q, q lye on the same side of the refracting surfaces, if the incident rays flow from Q, the refracted rays will also flow from q; and if the incident rays flow towards Q, the refracted will also flow towards q: and the contrary will happen when Q and q are on contrary sides of the refracting surfaces. Because the rays are continually going forwards.

The 211th and 212th articles are applicable to refractions as well as reflections.

CHAP-

Chapter IV.

To determine the place, magnitude and situation of images formed by refracted rays.

Proposition I.

244. Images *formed by refractions at plane surfaces are similar to the objects, and are always erect, or in a similar situation to the objects, and on the same side of the planes.*

Fig. 336, 337. Let PQR be an object radiating upon a refracting plane ACB; to which draw the perpendiculars PA, QC, RB, &c. and in these perpendiculars take Ap to AP, Cq to CQ, Br to BR in the given ratio of the sine of incidence to the sine of refraction[a]; and the focuses p, q, r, &c. will constitute a similar image in a similar situation to the object; the parts pq, qr being in the same ratio as PQ, QR. This is self-evident when the object is parallel to the refracting plane; and when it is inclined, produce it till it cuts the plane in D; and the produced image will also cut it in the same point D. For supposing the perpendicular BrR to move towards D, the lines BR, Br being in a given ratio will vanish both together: and because the triangle pDP is cut by parallel lines qQ, rR, it will be as pq to PQ (so qD to QD) so qr to QR[b]; and alternately pq to qr as PQ to QR. In like manner if the rays that belong to the focuses p, q, r be refracted again by another plane, either parallel or inclined to AB, their second focuses will constitute a second image, similar to the first image and consequently similar to the object; and so on. *Q. E. D.*

a Art. 223.

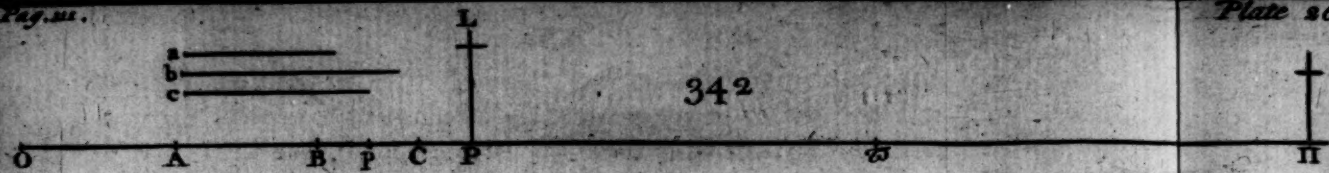
b Euc. VI. 2.

Proposition II.

Fig. 338 to 341. 245. *If an arch of a circle* PQR *described upon the center* E, *of a spherical surface, sphere or lens, be considered as an object, its image* pqr *will be a similar concentrick arch; whose length will be to the length of the object in the ratio of their distances from the common center* E; *and the image will be erect or inverted, with respect to the object, according as they lye on the same side of the center or on contrary sides.*

The proposition is evident by inspection of the 338th figure in all cases of refractions made by concentrick surfaces; because the parts of these surfaces are alike exposed to the parts of the concentrick object. And in a lens the focuses of all the pencils of parallel rays lye also in a concentrick arch GFH; whence Pp and Qq being third proportionals to two pair of equal distances PG and PE, QF and QE[c], are also equal; and so the image pqr is also a concentrick arch. Now since the axes of the

c Art. 239.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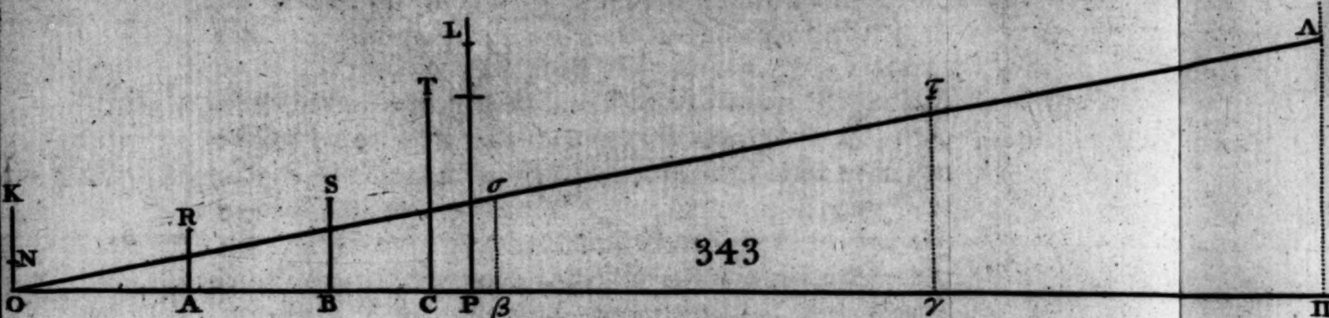
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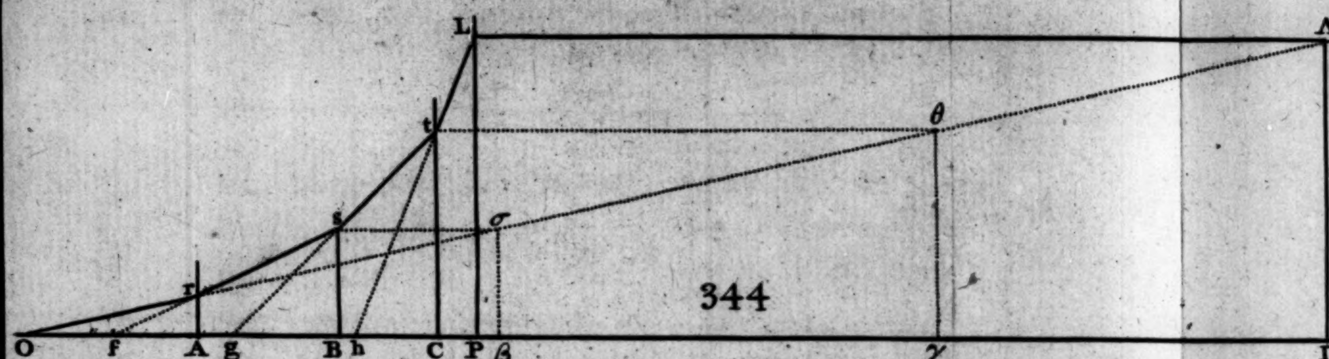
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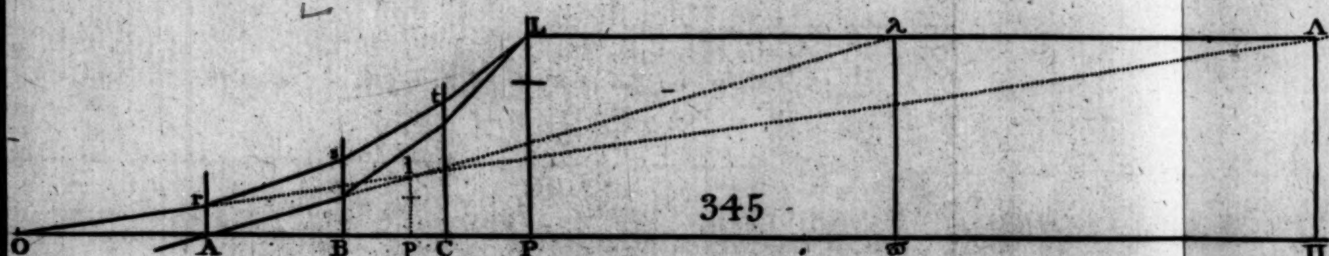
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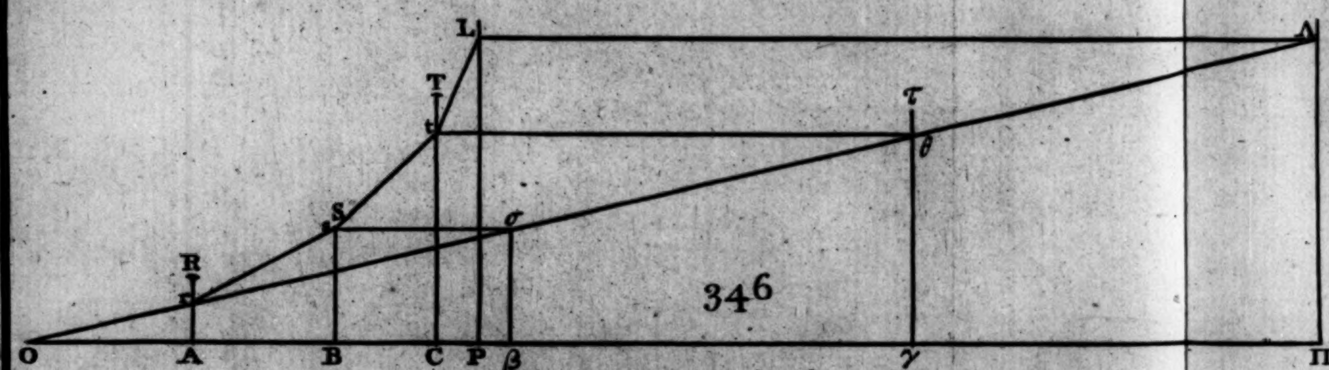
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the pencils are reckoned straight lines passing through E[a], the angles pEr, PER are equal; and therefore the ratio of the image to the object, is the same as of their distances from the center E. And according as their corresponding extremities P, p are on the same or on contrary sides of E, they lye on the same or contrary sides of their middle points Q, q; that is the image is accordingly erect or inverted. *Q. E. D.* a Art. 229.

246. *Corol.* The smaller the circular object is with respect to its radius or distance from the center E, the nearer it approaches in shape to a straight line, and so does its similar image. Consequently a small straight object placed at a good distance from the center of the glass may be reckoned to have a straight image very nearly[b]: though in strictness of geometry it is an arch b Art. 204. of a conick section determined in the Remarks. And by these propositions the images of all objects may be determined, by finding the images of their out-lines. Here I conclude the elementary propositions, which are also demonstrated algebraically in the Remarks.

CHAPTER V.

To determine the apparent distance, magnitude, situation, degree of distinctness and brightness, the greatest angle of vision and visible area, of an object seen by rays successively reflected from any number of plane or spherical surfaces; or successively refracted through any number of lenses of any sort, or through any number of different mediums whose surfaces are plane or spherical. With an application to Telescopes and Microscopes.

PROPOSITION I.

247. HAVING *the focal distances and apertures of any number of lenses of any sort, placed at any given distances from one another and from the eye and object, it is required to find the apparent distance, magnitude, situation, degree of distinctness and brightness of the object seen through all the lenses; together with the greatest angle of vision and visible area of the object, and the particular aperture which limits them both.*

248. Let PL be an object viewed by the eye at O through any num- Ap. distance. ber of lenses placed at A, B, C, whose focal distances are the lines a, b, c, Fig. 342. and whose common axis is the line $OABCP$. The distance OP may be considered as divided by the glasses at A, B, C, into two such parts as OA, AP; OB, BP; OC, CP; or into three such parts as OA, AB, BP; OA, AC, CP; or into four such parts as OA, AB, BC, CP; and

so on as far as the number of glasses permits. All the several products of such corresponding parts, applyed respectively to the focal distance, or to the product of the focal distances of the glasses, which are placed at the point or points of division, will give so many several lines; which must be looked upon as negative, if there be an odd number of convex glasses at the points of division, otherwise as affirmative. Let $O\Pi$ be the sum of OP and of those several lines according to their signs, and it will be the apparent distance of the object.

Ap. magnitude. a Art. 100.

249. And its apparent magnitude will be to its true [a] magnitude, as OP to $O\Pi$.

Ap. situation.

250. And if the value of $O\Pi$ be affirmative, the object will appear upright, otherwise inverted. Or to express the same things in other words, the object will appear through all the glasses at the same distance, of the same magnitude, and in the same situation, as it would appear to the naked eye viewing it from the distance $O\Pi$, supposing it placed upright at Π when $O\Pi$ is affirmative, and inverted when $O\Pi$ is negative.

Ap. distinctness.

251. When $O\Pi$ is affirmative place it before the eye, otherwise behind it; then let the eye be removed from O to A, that its distance from the next glass may vanish; and here let $A\varpi$ be the apparent distance of the object PL, to be found and placed by the same rules as $O\Pi$ was. Then let Ap be to $A\varpi$ as AO to the difference of $O\Pi$ and $A\varpi$, if they lye the same way from O and A, otherwise to their sum; and let the order of the points A, p, ϖ be the same as the order of the points O, p, Π; and from the situation of this point p, a judgment may be formed of the degree of distinctness with which the object will appear. Because the rays flowing from P, by passing through the glasses, will be disposed to fall upon the eye, in the same manner as if the glasses being removed, they tended from the point p, when it falls before the eye, or towards the point p, when it falls behind the eye.

Visual angle. Fig. 343.

252. Let the lines AR, BS, CT be the semidiameters of the given apertures of the glasses A, B, C; and let $O\beta$ be the apparent distance of the line BS, seen through the glass A; and $O\gamma$ the apparent distance of the line CT seen through the glasses A and B; to be found as above. Erect the perpendiculars $\beta\sigma$ equal to BS, and $\gamma\tau$ equal to CT, and then the least of the angles, which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$, subtends at O, will be half the greatest angle of vision.

Visible area.

253. Let this angle be $\beta O\sigma$, and let $O\sigma$, produced, cut a perpendicular to the axis at Π in Λ; then PL taken equal to $\Pi\Lambda$ will be the semidiameter of the greatest area of the object PL, that can be seen at one view from O, through the given apertures of all the glasses: and therefore PL or $\Pi\Lambda$, the semidiameter of the visible area, will be to $\beta\sigma$ or BS, the semidiameter of the aperture which limits it, as $O\Pi$ the apparent distance of the area, to $O\beta$ the apparent distance of that aperture.

254.

254. And by the supposition that $\beta O\sigma$ is the least of all the angles subtended at O by any of the given lines AR, $\beta\sigma$, $\gamma\tau$, it will follow that the aperture which limits the visual angle and visible area, belongs to the glass B. Where limited.

255. Since the magnitude of the pupil is subject to be varied by various degrees of light, let NO be its semidiameter when the object PL is viewed by the naked eye from the distance OP; and upon a plane that touches the eye at O let OK be the semidiameter of the greatest area, visible through all the glasses to another eye at P, to be found as PL was; or which is the same thing let OK be the semidiameter of the greatest area inlightened by a pencil of rays flowing from P through all the glasses; and when this area is not less than the area of the pupil, the point P will appear just as bright through all the glasses, as it would do if they were removed: but if the inlightened area be less than the area of the pupil, the point P will appear less bright through the glasses, than if they were removed, in the same proportion as the inlightened area is less than the pupil. And these proportions of apparent brightness would be accurate, if all the incident rays were transmitted through the glasses to the eye, or if only an insensible part of them were stopt. Ap. brightness.

DEMONSTRATION.

256. For let any ray $OrstL$, of a pencil supposed to flow from the eye to the object, belong successively to the focuses f, g, b, after its successive refractions at the glasses Ar, Bs, Ct; and then let it fall upon the object at L. The point L will therefore be seen by the ray $LtsrO$ returning back along the same lines to the eye at O[a]. Let $L\Lambda$ drawn parallel to OP meet the visual ray Or, produced, in Λ; and let the rectangle $PL\Lambda\Pi$ be compleated; and $O\Pi$ will be the apparent distance of the object[b]. Let us at first suppose all the lenses to be concaves; and since the triangles OAr, $O\Pi\Lambda$ are equiangular, we have OA to $O\Pi$ in the same ratio as Ar to $\Pi\Lambda$ or PL; or in a ratio compounded of Ar to Bs, Bs to Ct, Ct to PL; or compounded of fA to $fA+AB$, gB to $gB+BC$, bC to $bC+CP$; and consequently $O\Pi = OA \times \frac{fA+AB}{fA} \times \frac{gB+BC}{gB} \times \frac{bC+CP}{bC}$; by which theorem the apparent distance $O\Pi$ will be given so soon as fA, gB, bC can be found. These may be found by Art 239 as follows; $fA = \frac{OA \times a}{OA+a}$; $gB = \frac{\overline{fA+AB} \times b}{fA+AB+b}$; $bC = \frac{\overline{gB+BC} \times c}{gB+BC+c}$. Hence it is easy to collect, that if the eye at O views an object at B through one glass at A, its apparent distance $O\beta = OB + \frac{OAB}{a}$; that if the eye at O

Ap. distance. Fig. 344.

a Art. 11.

b Art. 139.

views an object at C, through two glasses at A, B, its apparent distance $O\gamma = OC + \frac{OAC}{a} + \frac{OBC}{b} + \frac{OABC}{ab}$; that if the eye at O views an object at P through three glasses at A, B, C, its apparent distance $O\Pi = OP + \frac{OAP}{a} + \frac{OBP}{b} + \frac{OCP}{c} + \frac{OABP}{ab} + \frac{OACP}{ac} + \frac{OBCP}{bc} + \frac{OABCP}{abc}$; and so on continually as the solution of the problem directs. Now if any of the lenses be convex, the focal distances of such lenses must be looked upon as negative, since they lye the contrary way to those of concave lenses, when the incident rays come the same way upon both sorts: therefore the terms which involve any odd number of convex lenses at the points of division, must be looked upon as negative.

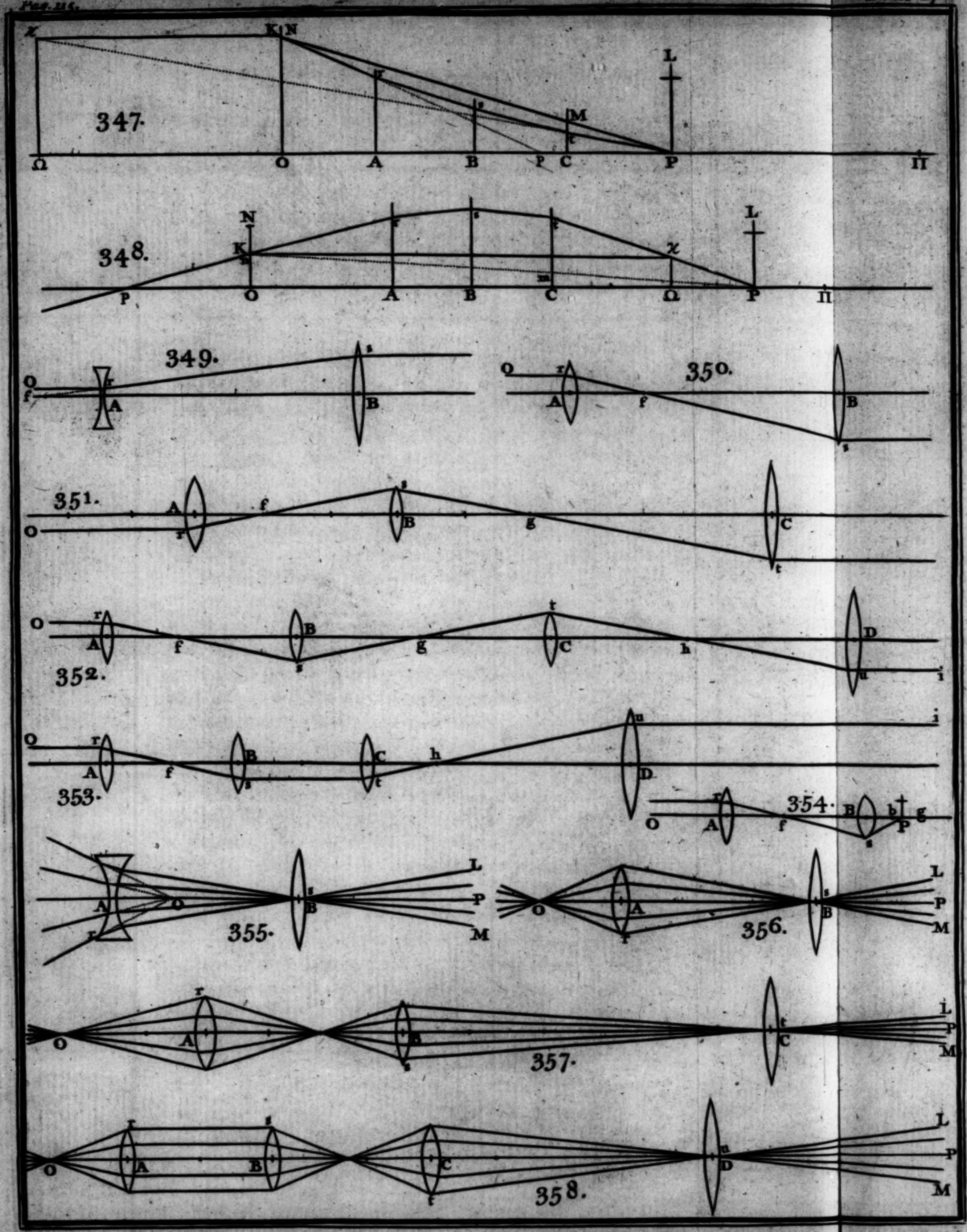
Ap. magnitude. The determination of the apparent magnitude is evident by Art. 141.

Ap. situation. And that of the apparent situation by the latter part of Art. 139.

Ap. distinctness. Fig. 345. 257. Compleat the rectangle $LP\varpi\lambda$ and join $A\lambda$ meeting $O\Lambda$ in l; and the line lp, drawn perpendicular to the axis of the glasses, will be the last image of the object LP. Because the same point L is seen by a ray which falls upon the eye at O in the direction ΛO, and also by a ray which falls upon the eye at A in the direction λA; and therefore the point l where these directions cross one another is the focus of the emergent rays. Now since the triangles Apl, $A\varpi\lambda$ and also Opl, $O\Pi\Lambda$ are equiangular, we have Ap to $A\varpi$ as (pl to $\varpi\lambda$ or $\Pi\Lambda$, or as) Op to $O\Pi$, or as $Op \mp Ap$ or OA to $O\Pi \mp A\varpi$, according as p falls without or within the line OA, and consequently according as $O\Pi$ and $A\varpi$ lye the same or contrary ways from O and A. And the order of the poins A, p, ϖ is the same as the order of the points A, l, λ, or of the points O, l, Λ, or of the points O, p, Π.

Visual angle. Fig. 343, 346. 258. Let $O\sigma$ cut the perpendiculars AR in r, $\gamma\tau$ in θ, $\Pi\Lambda$ in Λ, and compleat the rectangles $B\beta\sigma s$, $C\gamma\theta t$, $P\Pi\Lambda L$; then by the supposition that the angle $\beta O\sigma$ is the least of the angles, which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$ subtends at O, it follows that Ar is less than AR; and $\gamma\theta$ less than $\gamma\tau$, or Ct less than CT. But joining rs, st, tL, these lines will be described by a ray passing from O to L. Because the lines Or, $Or\sigma$, $Or\theta$, $Or\Lambda$ are the several apparent distances of the points r, s, t, L seen in one common direction Or. But in the construction $\beta\sigma$ was taken equal to BS; and supposing the visual angle $\beta O\sigma$ to be increased never so little, the equal perpendiculars $\beta\sigma$, Bs would also be increased; and then Bs would become bigger than BS, and consequently the outmost ray Lts would be stopt at s for want of a larger aperture than BS.

Visible area. 259. The perpendicular $\Pi\Lambda$ or PL would also be increased by increasing the angle $\beta O\sigma$, but this being impossible without increasing BS, it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it is plain that PL is the semidiameter of the greatest area that can be seen at one view from O through all the given apertures.

260. And so it is manifest that the vision is confined by that aperture at B, which appears through the intermediate glasses, under a lesser angle $\beta O \sigma$, than any other aperture would appear under, if the rest of the apertures were sufficiently inlarged for the purpose of seeing it. Where limited.

261. If OK be not less than ON, the area of the pupil will be totally inlightened by the pencil that flows from P. Let $PtsrN$ be a ray of that pencil, cutting the object-glass Ct in t; and supposing the glasses were removed, let an unrefracted ray PMN cut the line Ct in M. Then the quantity of refracted rays which fall upon the line NO, is to the quantity of unrefracted rays which would fall upon it, as the angle CPt to the angle CPM, that is as the apparent magnitude of the line NO, seen from P, to the true. And therefore by turning the figure round about the axis OP, the quantity of refracted rays which fill the pupil, is to the quantity of unrefracted rays which would fill it (as the apparent magnitude of any surface at O seen from P, to the true; or as the apparent magnitude of any surface at P seen from O to the true[a]; and consequently) as the apparent magnitude of the least surface or physical point P, to the true; that is as the picture of the point P formed upon the retina by those refracted rays, to its picture formed by the unrefracted rays. These pictures of the point P are therefore equally bright and cause the appearance of P to be equally bright in both cases. Now let the pupil be larger than the greatest area inlightened at O by the pencil that flows from P; and supposing a smaller pupil equal to this area, we have shewn that the pictures of P upon the retina made by refracted and unrefracted rays would be equally bright; and consequently each of them would be less bright than when the larger pupil is filled with unrefracted rays, in the same proportion as the smaller pupil, or area inlightened by the refracted rays, is less than the larger pupil, inlightened by unrefracted rays. Hitherto I have supposed the picture of the point P to be distinct upon the retina, or proportionable to the angle which measures the apparent magnitude of P; let us now suppose it to be confused and the conclusion will not be altered. For the confused picture of P diffuses an equal portion of its own light upon every point (equidistant from its center,) quite round its distinct bounds, and receives from every point as many such portions of their lights, which came from other points of the object. Ap. brightness. Fig. 347. Fig. 348.

a Art. 262.

262. *Corol.* 1. While the glasses are fixt, if the eye and object be supposed to change places, the apparent distance, magnitude and situation of the object will be the same as before. For the interval OP being the same, and being divided by the same glasses into the same parts, will give the same theorem for the apparent distance as before[b]; namely PO

b Art. 248.

$$+\frac{PCO}{c}+\frac{PBO}{b}+\frac{PAO}{a}+\frac{PCBO}{cb}+\frac{PCAO}{ca}+\frac{PBAO}{ba}+\frac{PCBAO}{cba}.$$

Fig. 347, 348. 263. *Corol.* 2. When an object PL is seen through any number of glasses, the breadth of the principal pencil where it falls on the eye at O, is to its breadth at the object-glass C, as the apparent distance of the object, to its real distance from the object-glass; and consequently in Telescopes, as the true magnitude of the object, to the apparent; that is OK is to Ct as $O\Pi$ to PC. For let $K\varkappa$ drawn parallel to the axis OP, cut Pt produced in $\varkappa$, and compleat the rectangle $\varkappa KO\Omega$, then $P\Omega$ is the apparent distance of an object OK seen from P through all the glasses; and the triangles $P\Omega\varkappa$, PCt being equiangular, we have OK or $\Omega\varkappa$ to Ct as $P\Omega$ to PC, or as $O\Pi$ to PC by the foregoing corollary.

264. *Corol.* 3. When the rays flowing from P through all the glasses, fall perpendicularly upon a fixt plane at O, their density will be uniform in every part of the inlightened area. For supposing all the incident rays to be transmitted, their quantity in the areas at C and at O will be the same; and this quantity being uniformly dense in the area at C, is as this area, or as the area at O (the ratio of these areas being invariable[a]) and consequently is uniformly dense in the area at O. And whatever part of the light may not be transmitted to the area at O; yet equal portions of it will be reflected back from equal portions of any of the surfaces of the glasses very nearly; (because all the rays fall almost perpendicularly upon every surface;) and consequently equal portions of the rays will be intercepted from falling upon equal portions of the area at O.

a Art. 263.

265. *Corol.* 4. This uniform density of the refracted rays in the area at O, is to the uniform density of unrefracted rays, which would fall upon it, if the glasses were removed and which come from the same point P, as the apparent magnitude of the point P, or of any surface at P, to the true; supposing all the rays were transmitted. This will appear from the first part of the 261st article.

266. *Corol.* 5. If the quantity of incident light, in passing through the glasses, be diminished in no greater a proportion than that of the greatest aperture of the pupil to the given aperture ON, and the inlightened area be not less than the greatest aperture; the pupil will dilate it self till it receives the same quantity of light as in vision with the naked eye[b]. But when the inlightened area is less than the given aperture of the pupil, the natural brightness of the object will appear diminished in the glasses, in a ratio compounded of the ratio of the aperture of the pupil, to the inlightened area[c], and of the quantity of incident light to the quantity of emergent light.

b Art. 264.

c Art. 255.

267. *Corol.* 6. It is evident that an object seen through glasses may appear as bright as to the naked eye; but never brighter, even though all the incident light be transmitted through the glasses.

268.

268. *Corol.* 7. The glasses and object being fixt, the apparent brightness of the point P, seen by refracted rays, is invariable wherever the eye is placed, while the pupil is filled with rays that come from that point; but where it is not filled, the apparent brightness varies directly in a duplicate ratio of, Op, the distance of the pupil from the last image of P. For the density of the rays, and the apparent magnitude of the point P, and consequently the magnitude of its picture upon the retina, do all vary reciprocally in a duplicate ratio of Op [a]. Consequently while the pupil does not vary and is filled with rays, the quantity that enters it is as the area of the picture of P upon the retina; whose brightness is therefore invariable: but where the pupil is not filled with rays, the quantity that enters the pupil is invariable, while the area of the picture varies reciprocally in a duplicate ratio of Op, and consequently while its brightness varies directly in the duplicate ratio of Op. And this is so whatever part of the incident light be stopt by the glasses. Fig. 348.

a Art. 58. 106. 111.

269. *Corol.* 8. When the object is so remote that the distances OP, AP, BP, CP, may be considered as equal to one another, then the apparent distance $O\Pi = OP$ into $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$.

270. *Corol.* 9. Hence when O and b are conjugate focuses of a pencil of rays refracted through any number of lenses A, B, C, the ratio of the angles AOr, Cbt, made by the incident and emergent parts of any ray with the axis of the glasses, is the same as of 1 to $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$. For this latter ratio is the same as the ratio of OP to $O\Pi$ by cor. 8; that is of the apparent magnitude of a remote object seen from O, to its true magnitude seen from O or b by the naked eye, or as the angle at O to the angle at b. Fig. 344.

271. *Corol.* 10. Hence if O be the focus of incident rays, the focus of the emergent rays from the last glass C, may be found by taking Cb to $O\gamma$ or $OC + \frac{OAC}{a} + \frac{OBC}{b} + \frac{OABC}{ab}$ as 1 to $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$, and by placing Cb contrary to the course of the rays if the second and third terms of this proportion have like signs, otherwise according to their course. For compleating the rectangle $\gamma Ct\theta$, Cb is to $O\gamma$ as the angle $\gamma O\theta$ to the angle Cbt [b], that is in the ratio abovementioned, by cor. 9.

b Art. 60.

272. *Corol.* 11. When the focal distances of the glasses, and the distances of the glasses from one another and from the object are such that 1

$+\frac{AP}{a}+\frac{BP}{b}+\frac{CP}{c}+\frac{ABP}{ab}+\frac{ACP}{ac}+\frac{BCP}{bc}+\frac{ABCP}{abc}=o$, the rays of any pencil will fall upon the eye in parallel lines; and then the apparent distance $O\Pi$ will be equal to $A\varpi=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}$ or $=-a$ into $1+\frac{BP}{b}+\frac{CP}{c}+\frac{BCP}{bc}$: And this apparent distance being invariable, the apparent magnitude, situation, and degree of distinctness and brightness will also be invariable wherever the eye is placed. For the rays flowing from L will fall parallel upon the eye when Ol and Al, or $O\lambda$ and $A\lambda$ are parallel; and consequently when $O\Pi=A\varpi$, or $O\Pi-A\varpi=o$. Now by putting $OA=o$ in the value of $O\Pi=OP+\frac{OAP}{a}+\frac{OBP}{b}+\frac{OCP}{c}+\frac{OABP}{ab}+\frac{OACP}{ac}+\frac{OBCP}{bc}+\frac{OABCP}{abc}$, we have $A\varpi=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}$; which being taken from $O\Pi$, there remains $O\varpi-A\varpi=1+\frac{AP}{a}+\frac{BP}{b}+\frac{CP}{c}+\frac{ABP}{ab}+\frac{ACP}{ac}+\frac{BCP}{bc}+\frac{ABCP}{abc}=o$; which gives $-a$ into $1+\frac{BP}{b}+\frac{CP}{c}+\frac{BCP}{bc}=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}=A\varpi=O\Pi$.

Fig. 345.

273. *Corol.* 12. Hence when the object is so remote, that the distances OP, AP, BP, CP may be considered as equal to one another, the rays which fall parallel upon the first glass will emerge parallel from the last, if the glasses be so disposed that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=o$; and on the contrary: and then the apparent distance $O\Pi=(A\varpi=)\ OP$ into $1+\frac{AB}{b}+\frac{AC}{c}+\frac{ABC}{bc}$ or $=-a\times OP$ into $\frac{1}{b}+\frac{1}{c}+\frac{BC}{bc}$. And consequently in two concave lenses A, B, the apparent magnitude is to the true, or OP to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b}$; in three concave lenses A, B, C, as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{BC}{bc}$; and in four concave lenses A, B, C, D, as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{BC}{bc}+\frac{BD}{bd}+\frac{CD}{cd}+\frac{BCD}{bcd}$. The unit is rejected in these deductions from cor. 11. as being inconsiderable in comparison to the distance of the object.

274. *Corol.* 13. Since the eye, the glasses and object are placed in a given order, their intervals OA, AB, BC, &c. must be considered as being all affirmative, and since every term in this equation, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

+

$+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=o$, for placing three glasses A, B, C, as abovementioned, is affirmative; the sum of the terms cannot be nothing, and consequently the rays cannot fall upon the eye in parallel lines, unless some one of the focal distances be negative, or some one of the glasses be convex. Now in a Telescope supposed to consist of two concave lenses A, B, we have $\frac{1}{a}+\frac{1}{b}+\frac{AB}{ab}=o$, or $AB=-a-b$; which shews that AB the interval of the two glasses, must be equal to the sum or difference of their focal distances, according as both are convex, or one convex and the other concave. In the first case we have OP to $O\Pi$ as $\frac{1}{a}$ to $-\frac{1}{b}$ or as b to $-a$, by cor. 12, where the negative value of $O\Pi$ shews that the object will appear inverted[a]. In the second case we have OP to $O\Pi$ as b to a, which shews that the object will appear upright. Fig. 349, 350.

[a] Art. 250.

275. *Corol.* 14. For placing three concave glasses as above, we had $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=o$*, or $b+a+\frac{ab}{c}+AB+\overline{AB+BC}\times\frac{b}{c}+\frac{aBC}{c}+\frac{ABC}{c}=o$, or $\overline{AB+a+b},c+\overline{AB+a},b+\overline{AB+a+b},BC=o$, or $\overline{AB+a+b},\ \overline{BC+c}+\overline{AB+a},b=o$. Let all the glasses be convex then $AB-a-b$ is to $AB-a$ as b to $BC-c$; by which proportion either of the intervals AB, BC will be given when the other is assumed as most convenient. We had OP to $O\Pi$ as $\frac{bc}{a}$ to $BC-b-c$*; where if BC be affirmative or bigger than $b+c$ the object will appear upright[b]. Put $BC-b-c=b$, or $BC=2b+c$, and the object will appear upright and magnified in the ratio of OP to $O\Pi$ or of c to a, whatever be the length of b. To determine the other interval AB, we found $AB-a-b$ to $AB-a$ as b to $BC-c$ or $2b$ by the assumption; consequently $2AB-2a-2b=AB-a$, and $AB=a+2b$. Hence if we put $b=a$, then $AB=3a$, and $BC=2a+c$.

* Art. 273. Fig. 351. * Art. 273. [b] Art. 250.

276. *Corol.* 15. But for the rays of a pencil to emerge parallel to one another from any number of glasses, it is only necessary that their last focus should coincide with the principal focus of the last glass; as is evident by conceiving the emergent rays to return backwards in the same parallel lines. Therefore all the intervals of the glasses but the last may be assumed as shall be found most convenient for other purposes. And then if any point O be the focus of incident rays their successive focuses f, g, h, i, &c. after their refractions at A, B, C, D, &c. may be easily found by these rules[c]; $fA=\frac{OA,a}{OA+a}$; $gB=\frac{\overline{fA+AB},b}{fA+AB+b}$; $hC=\frac{\overline{gB+BC},c}{gB+BC+c}$; $iD=\frac{\overline{hC+CD},d}{hC+CD+d}$; &c.

Fig. 352, 353. [c] Art. 256.

&c. taking due care to observe the signs of fA, gB, hC, &c. and to place them forwards or according to the course of the rays when negative, and backwards when affirmative. For example in a Telescope consisting of four convex glasses, supposing rays to come parallel upon the eye-glass A, the line AO must be made infinite and consequently $fA = -a$. Hence $gB = \frac{-a + AB}{-a + AB - b} \times -b$, which being made infinite, that the rays may go parallel between the glasses A, B, gives $-a + AB - b = o$ or $AB = a + b$. Hence whatever be the interval BC, we have $hC = -c$, and consequently $iD = \frac{-c + CD}{-c + CD - d} \times -d$, which being made infinite, that the rays may emerge parallel, gives $-c + CD - d = o$, or $CD = c + d$. Now when the four glasses were concaves the ratio of the apparent magnitude to the true, or of OP to $O\Pi$, was $-\frac{1}{a}$ to $\frac{1}{b} + \frac{1}{c} +$

a Art. 273. $\frac{1}{d} + \frac{BC}{bc} + \frac{BD}{bd} + \frac{CD}{cd} + \frac{BCD}{bcd}$ [a], or $-\frac{bcd}{a}$ to $D\overline{C + d + c}$, $C\overline{B + b + D\overline{C + d}, c}$; by a like reduction to that in cor. 14; or in four convex glasses, as $-\frac{bcd}{a}$ to $D\overline{C - d - c}$, $C\overline{B - b + D\overline{C - d}, -d}$; or because $CD = c + d$, as $-\frac{bcd}{a}$ to $-cc$, or as db to ca; or putting $b = a$, as d to c, whatever be the focal distance of the equal glasses A, B. And the affirmative value of $O\Pi$ shews that the object will appear upright [b].

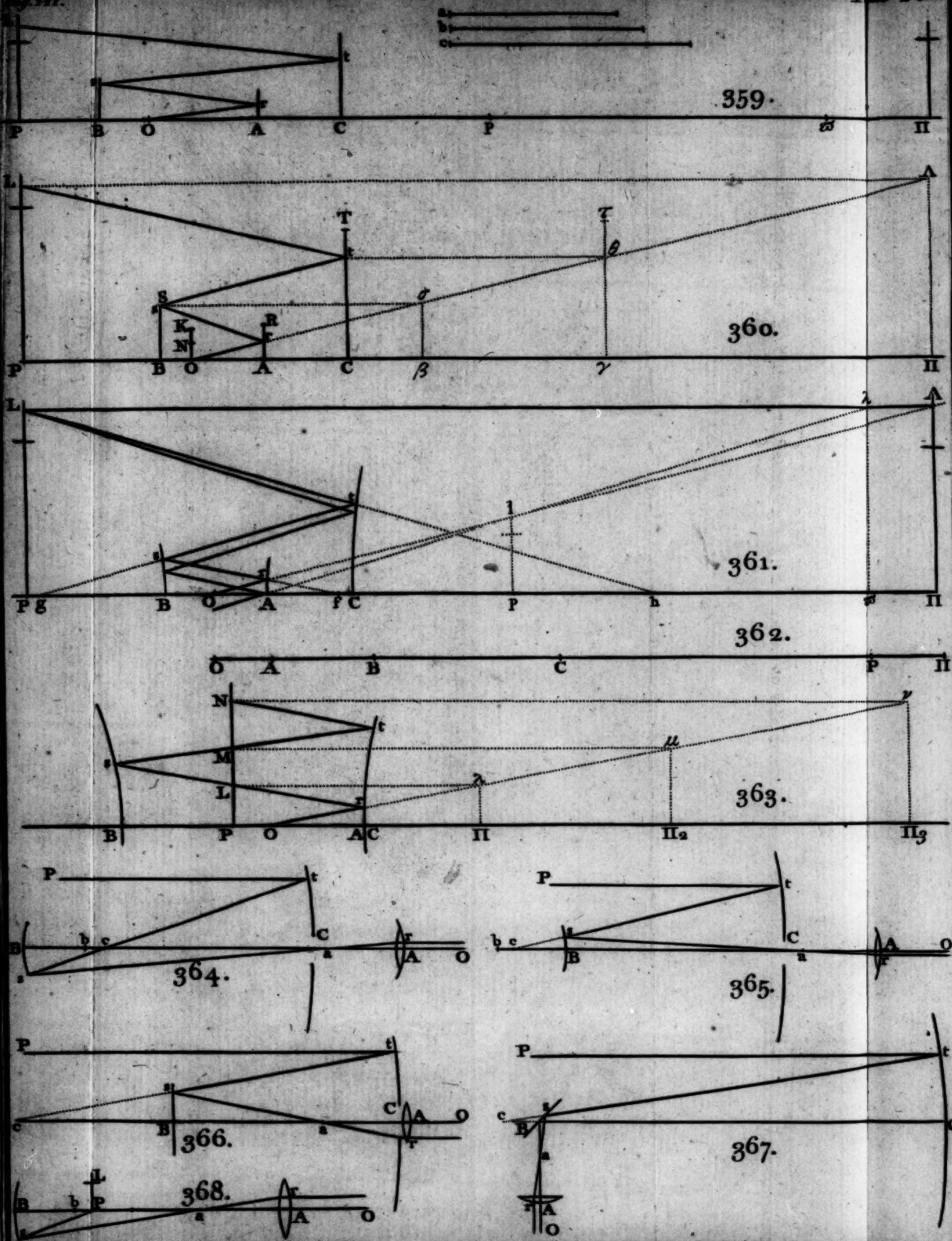
b Art. 250.

Fig. 354. c Art. 276.

277. *Corol.* 16. In a microscope composed of two convex glasses A and B, if the object PL be placed at g to be found as before [c], the rays will return to the eye in parallel lines; and then, by cor. 11. the apparent distance $O\Pi = a \times \overline{1 - \frac{BP}{b}} = \frac{a}{b} \times \overline{b - BP} = -\frac{a}{b} \times Pb$, taking $Bb = b$. And so the apparent magnitude is to the true, or OP to $O\Pi$, as OP to $-\frac{a}{b} Pb$, that is in a ratio compounded of b to a and of OP to Pb: and the negative value of $O\Pi$ shews that the object will appear inverted.

278. *Corol.* 17. Hence again when the object is remote, as in a telescope, the apparent magnitude is to the true as b to a; because the ratio of OP to Pb becomes a ratio of equality.

279. *Corol.* 18. Since the area of an object-glass is the common base of all the pencils that flow from the several points of an object, whether near or remote; the middle ray of every pencil will pass straight through the middle point of this glass. This middle point may therefore be considered as the focus from whence the middle rays flow upon the subsequent glass or glasses; and consequently if these rays emerge from the last glass converging to a focus, and the pupil of the eye be placed at this focus, it will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will receive all the middle rays though it were contracted to a point; and when it is more open it will also receive as many collateral rays of every pencil as its aperture can admit, and sometimes the whole pencils. The visible area of the object will therefore be the greatest when the eye is at this focus; for in moving the pupil either way from this focus, till it comes to a place in the pencil of middle rays, where its section is as large as the pupil, the outmost of these rays will begin to be excluded from the pupil; and then the visible area of the object begins to be contracted. And in like manner if the middle rays shall emerge diverging from a focus placed beyond the eye-glass, the outmost of them will be gradually excluded from the pupil while it recedes from that focus or from the eye-glass; and consequently the visible area will be greatest in this case when the pupil is close to the eye-glass. Now this focus O of emergent rays, when the center of the object-glass is the focus of the incident rays, may be found several ways. By art. 271, if B be the focus of incident rays and the glass A be concave, $AO = \frac{AB}{1 + \frac{AB}{a}}$; and if C be the focus of incident rays and the glasses B, A be concaves, $AO = \frac{AC + \frac{ABC}{b}}{1 + \frac{AB}{a} + \frac{BC}{b} + \frac{ABC}{ab}}$; and so on; and if any of the glasses be convex the signs of their focal distances must be changed. For example in *Galileo's* telescope where the glass A is concave and $AB = b - a$, we have $AO = a - \frac{aa}{b}$, where a being bigger than $\frac{aa}{b}$, AO is affirmative and consequently lyes the contrary way from A to the course of the rays [a], which therefore emerge from A diverging from O; and so the visible area will be the greatest when the eye is put close to the glass A. In the astronomical telescope $AO = -a - \frac{aa}{b}$, by making the line a negative upon account of the convex eye-glass. Here the point O lyes on the outside of the telescope a little farther from the eye-glass than its principal focus, the small overplus $\frac{aa}{b}$ being to a as a to b. In the telescope consisting of two convex eye-glasses so placed that $AB = a + 2b$ and $BC = 2b + c$ [b], and consequently $AC = a + 4b + c$; by substituting these values in the rule above, we have $AO = -a - \frac{aa}{b} - \frac{aa}{c}$; or when $b = a$, $AO = -2a - \frac{aa}{c}$. Here the place of the eye is not much farther from A than twice its focal distance.

Fig. 355 to 358.

Fig. 355.

a Art. 270.

Fig. 356.

Fig. 357.

b Art. 275.

280. *Corol.* 19. The last image pl is to the object PL, as Op, the distance of the image from the eye, to $O\Pi$, the apparent distance of the object. For the triangles plO, $\Pi\Lambda O$ are similar, and $\Pi\Lambda$ is equal to PL. Fig. 345.

LEMMA.

281. Imagine a pencil of rays after ſeveral ſucceſſive reflections and refractions at ſeveral ſurfaces, to belong to ſeveral ſucceſſive focuſes, as in teleſcopes; then if any part of this pencil be ſtopt by an obſtacle of any ſhape and in any place, the other part that is not ſtopt will belong to the ſame ſucceſſive focuſes as the whole did before. Conſequently when ſeveral ſucceſſive images are formed by the ſucceſſive focuſes of ſeveral pencils, their places, ſhapes and magnitudes will continue the ſame, after any parts of thoſe pencils are ſtopt, as they were before. Therefore to determine the focuſes and images formed by ſuch partial pencils, we may juſtly argue upon any lines of a pencil along which rays might paſs, as if they really did paſs, or as if theſe lines had the properties of rays; and all the concluſions will be the ſame in both caſes, excepting thoſe concerning apparent brightneſs.

PROPOSITION II.

282. *Having the focal diſtances and apertures of any number of reflecting ſurfaces, either concave or convex, placed at any diſtances from each other and from the eye and object; it is required to find the apparent diſtance, magnitude, ſituation, degree of diſtinctneſs and brightneſs of the object ſeen by rays ſucceſſively reflected from all the ſurfaces; together with the greateſt angle of viſion and viſible area of the object, and the particular ſurface whoſe aperture limits them both.*

Ap. diſtance. Fig. 359.

283. Let the object PL be ſeen by rays, which in returning back from the eye at O to the object at P, are ſucceſſively reflected from the ſpherical ſurfaces A, B, C, whoſe focal diſtances are the lines a, b, c and common axis is the line $OABCP$. Take a line $O\Pi = OA + AB + BC + CP + \frac{OA, \overline{AB+BC+CP}}{a} + \frac{\overline{OA+AB}, \overline{BC+CP}}{b} + \frac{\overline{OA+AB+BC}, CP}{c} + \frac{OAB, \overline{BC+CP}}{ab} + \frac{OA, \overline{AB+BC}, CP}{ac} + \frac{\overline{OA+AB}, BCP}{bc} + \frac{OABCP}{abc}$; and let the terms that are applyed to the focal diſtances of any odd number of concave ſurfaces be looked upon as negative, otherwiſe as affirmative; and the line $O\Pi$ will be the apparent diſtance of the object.

Ap. magnitude and ſituation.

284. And its apparent magnitude will be to the true as OP to $O\Pi$. And if the value of $O\Pi$ be affirmative the object will appear upright, otherwiſe inverted.

Ap. diſtinctneſs.

285. When $O\Pi$ is affirmative place it before the eye, otherwiſe behind it; then let the eye be removed from O to A, that its diſtance from the next ſurface may vaniſh; and here let $A\varpi$ be the apparent diſtance of the object PL to be found and placed as above directed; then let Ap be to $A\varpi$

as

as AO to the difference of $O\Pi$ and $A\varpi$ if they both lye the ſame way from O and A, otherwiſe to their ſum; and let the order of the points A, p, ϖ be the ſame as of the points O, p, Π; and from the ſituation of this point p a judgment may be formed of the degree of diſtinctneſs with which the object will appear. For the rays flowing from the point P, will be diſpoſed, after all the reflections, to fall upon the eye tending from the point p, when it is before the eye, or towards the point p, when it is behind the eye.

286. Let the lines AR, BS, CT be the ſemidiameters of the given apertures of the ſurfaces A, B, C, and let $O\beta$ be the apparent diſtance of the line BS ſeen by reflection from the glaſs A, and $O\gamma$ the apparent diſtance of the line CT ſeen by reflections from the glaſſes B and A, to be found as above; erect the perpendiculars $\beta\sigma$ equal to BS, and $\gamma\tau$ equal to CT; and then the leaſt of the angles which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$ ſubtends at O, will be half the greateſt angle of viſion. Viſual angle. Fig. 360.

287. Let this viſual angle be $\beta O\sigma$, and let $O\sigma$ produced cut a perpendicular to the axis at Π in Λ, and PL taken equal to $\Pi\Lambda$ will be the ſemidiameter of the greateſt area of the object that can be ſeen at one view by the eye at O: and therefore $\Pi\Lambda$ the ſemidiameter of this viſible area, will be to $\beta\sigma$ or BS, the ſemidiameter of the aperture which limits it, as $O\Pi$ the apparent diſtance of that area, to $O\beta$ the apparent diſtance of that aperture. Viſible area.

288. And by the ſuppoſition that $\beta O\sigma$ is the leaſt of all the angles ſubtended at O by the given lines AR, $\beta\sigma$, $\gamma\tau$, it will appear that B is the glaſs whoſe aperture confines the viſion. Where limited.

289. The determination of the apparent brightneſs of the point P is alſo the ſame as in the foregoing propoſition, and may be deſcribed in this other manner. If another eye at P can ſee the whole pupil of the eye at O or more by reflection from the ſame glaſſes, the point P will appear as bright to the eye at O as if the glaſſes were removed; but if the eye at P can ſee but a part of the pupil at O, the point P will appear leſs bright to the eye at O, than before, in the ſame proportion as the viſible part of the pupil is leſs than the whole; ſuppoſing no part of the incident rays to be intercepted or loſt by the reflections, or but an inſenſible one. Now the viſible area of the eye at O ſeen from the point P may be determined as above. Ap. brightneſs.

290. The demonſtration of this propoſition is juſt the ſame as that of the foregoing propoſition, only by changing the words concave lenſes for convex ſurfaces; and refractions for reflections; and by referring to the 207th article inſtead of the 239th; and if the intervals OA, AB, BC, CP, taken according to the courſe of the rays, be joined together in one continued line $OABCP$, it appears by inſpection of the theorem for the apparent diſtance, that its expreſſion by the parts of this continued line Fig. 361, 362.

will be the very same as that of the theorem for lenses; namely $O\Pi = OP + \frac{OAP}{a} + \frac{OBP}{b} + \frac{OCP}{c} + \frac{OABP}{ab} + \frac{OACP}{ac} + \frac{OBCP}{bc} + \frac{OABCP}{abc}$. The corollaries in the former proposition concerning apparent brightness are also applicable to this proposition.

291. *Corol.* 1. If any of the reflecting surfaces be planes, they may be considered as portions of spherical surfaces whose diameters and focal distances are infinite; and then the terms in the value of the apparent distance, which are applyed to these infinite focal distances, will vanish. Thus if the surface B be a reflecting plane, then $O\Pi = OA + AB + BC + CP + \frac{OA,\overline{AB+BC+CP}}{a} + \frac{\overline{OA+AB+BC},CP}{c} + \frac{OA,\overline{AB+BC},CP}{ac}$. And if the surfaces A, B and C be all planes, $O\Pi = OA + AB + BC + CP$; which is the sum of all the lines described by the reciprocal motion of the nearest ray to the axis, in passing between the eye and object.

Fig. 363. 292. *Corol.* 2. When rays in flowing from the eye, fall several times upon an object $PLMN$, after several reflections from two surfaces A, B, the object will appear at as many several distances. As if the surfaces A and B be both convex, then after one reflection at A, $O\Pi = OA + AP + \frac{OAP}{a}$; and after two reflections at A and B, $O\Pi 2 = OA + AB + BP + \frac{OA,\overline{AB+BP}}{a} + \frac{\overline{OA+AB},BP}{b} + \frac{OABP}{ab}$; and after three reflections at A, B and A, $O\Pi 3 = OA + AB + BA + AP + \frac{OA,\overline{AB+BA+AP}}{a} + \frac{\overline{OA+AB},\overline{BA+AP}}{b} + \frac{\overline{OA+AB+BA},AP}{a} + \frac{OAB,\overline{BA+AP}}{ab} + \frac{OA,\overline{AB+BA},AP}{aa} + \frac{\overline{OA+AB},BAP}{ba} + \frac{OABAP}{aba}$; and so on. And by the foregoing corollary it appears what the apparent distances will be when one or both the reflecting surfaces are planes. And it is easy to be understood that one side of the object will be seen after every odd number of reflections, and its opposite side after every even number.

293. *Corol.* 3. By this and the foregoing proposition it is also evident that all the appearances of an object will be determinable by the same general rules, when seen by rays which in some places of their passage are reflected from surfaces of any sort, and in others are refracted through lenses of any sort. That is if any of the lines a, b or c be the focal distance of a concave or convex lens placed at A, B or C instead of a convex or concave surface, the general theorem for the apparent distance and every thing else will continue the same as before; excepting the subsequent course of the rays. The following corollaries are evident by the demonstrations of the like corollaries in the first proposition.

294.

294. *Corol.* 4. While the surfaces are fixt if the eye and object be supposed to change places, the apparent distance, magnitude and situation of the object will be the same as before.

295. *Corol.* 5. When the distances of the object from the eye and the surfaces are incomparably greater than their distances from one another, then $O\Pi = OP$ into $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$.

296. *Corol.* 6. When O and b are conjugate focuses of a pencil of rays successively reflected from any number of spherical surfaces A, B, C, the ratio of the angles AOr, Cbt, which any ray makes with the common axis of the surfaces before the first and after the last reflection, is the same as of 1 to $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$. Fig. 361.

297. *Corol.* 7. Having O the focus of incident rays to find b their conjugate focus after reflections from any number of spherical surfaces A, B, C; take Cb to $OA + AB + BC + \frac{OA,\overline{AB+BC}}{a} + \frac{\overline{OA+AB},BC}{b} + \frac{OABC}{ab}$, (the apparent distance of the last surface C,) as 1 to $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$ and observing the rule in the proposition for the sign of every line, place Cb from C contrary to the course of the rays reflected from C, if the second and third terms of the proportion have like signs, otherwise according to that course, and b will be the conjugate focus to O.

298. *Corol.* 8. Hence if the reflecting surfaces A, B, C be all planes, $Cb = OA + AB + BC$, and lyes contrary to the course of the rays reflected from C.

299. *Corol.* 9. When the object and the surfaces A, B, C are placed at such distances from one another, that the rays in any pencil shall fall upon the eye in parallel lines, then the apparent distance $O\Pi = -a$ into $1 + \frac{BC+CP}{b} + \frac{CP}{c} + \frac{BCP}{bc}$ or $= AB + BC + CP + \frac{AB,\overline{BC+CP}}{b} + \frac{\overline{AB+BC},CP}{c} + \frac{ABCP}{bc}$. And this apparent distance and consequently the apparent magnitude, situation, distinctness and brightness of the object will be invariable wherever the eye is placed.

300. *Corol.* 10. When the rays of a pencil fall upon the eye in parallel lines, the object and surfaces A, B, C are placed at such intervals that $1 + \frac{AB+BC+CP}{a} + \frac{BC+CP}{b} + \frac{CP}{c} + \frac{AB, \overline{BC+CP}}{ab} + \frac{\overline{AB+BC}, CP}{ac} + \frac{BCP}{bc} + \frac{ABCP}{abc} = 0$; and consequently when the object is remote, the surfaces are placed at such intervals that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{AB}{ab} + \frac{AB+BC}{ac} + \frac{BC}{bc} + \frac{ABC}{abc} = 0$; and on the contrary.

301. *Corol.* 11. In a telescope supposed to consist of two convex surfaces A and B, the apparent magnitude of the object, is to the true, or OP is to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b}$; and in a telescope consisting of three such surfaces A, B, C, OP is to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b} + \frac{1}{c} + \frac{BC}{bc}$; and in four such surfaces A, B, C, D, $O\Pi$ is to OP as $-\frac{1}{a}$ to $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{BC}{bc} + \frac{BC+CD}{bd} + \frac{CD}{cd} + \frac{BCD}{bcd}$; and so on.

Fig. 364, 365. 302. *Corol.* 12. In the figures for reflecting telescopes, let the points a, b, c be the principal focuses of the respective given surfaces A, B, C; and when the rays in any one pencil are parallel before the first and after the last reflections at A and C, the points a, c must be conjugate focuses with respect to the intermediate reflection of the same rays at the surface B. Therefore if the interval AB and consequently the interval ab be assumed as most convenient, say as ba to bB so bB to bc and the point c will be determined[a] and consequently the point C, and also the interval BC. Now if all the surfaces be concaves, the apparent magnitude is to the true, as $\frac{1}{a}$ to $-\frac{1}{b} - \frac{1}{c} + \frac{BC}{bc}$*, or as $\frac{bc}{a}$ to $BC - b - c$ or bc the interval of the focuses b and c; that is in a ratio compounded of c to a and of b to the interval bc; which being affirmative shews that the object will appear upright[b]. But if the surface B be convex and only A and C be concaves, the apparent magnitude will be to the true as $\frac{1}{a}$ to $\frac{1}{b} - \frac{1}{c} - \frac{BC}{bc}$, or as $\frac{b,c}{a}$ to $c - b - BC$ or $-bc$, the interval of the focuses b and c; that is in a ratio compounded of c to a and of b to bc, the interval of b and c; which being negative shews that the object will appear inverted. And if the interval bc, or the ratio of $\frac{b,c}{a}$ to the interval bc, be assumed at first, then the places of A and a will be determined by making bc, bB, ba continual proportionals.

a Art. 207.

* Art. 301.

b Art. 285.

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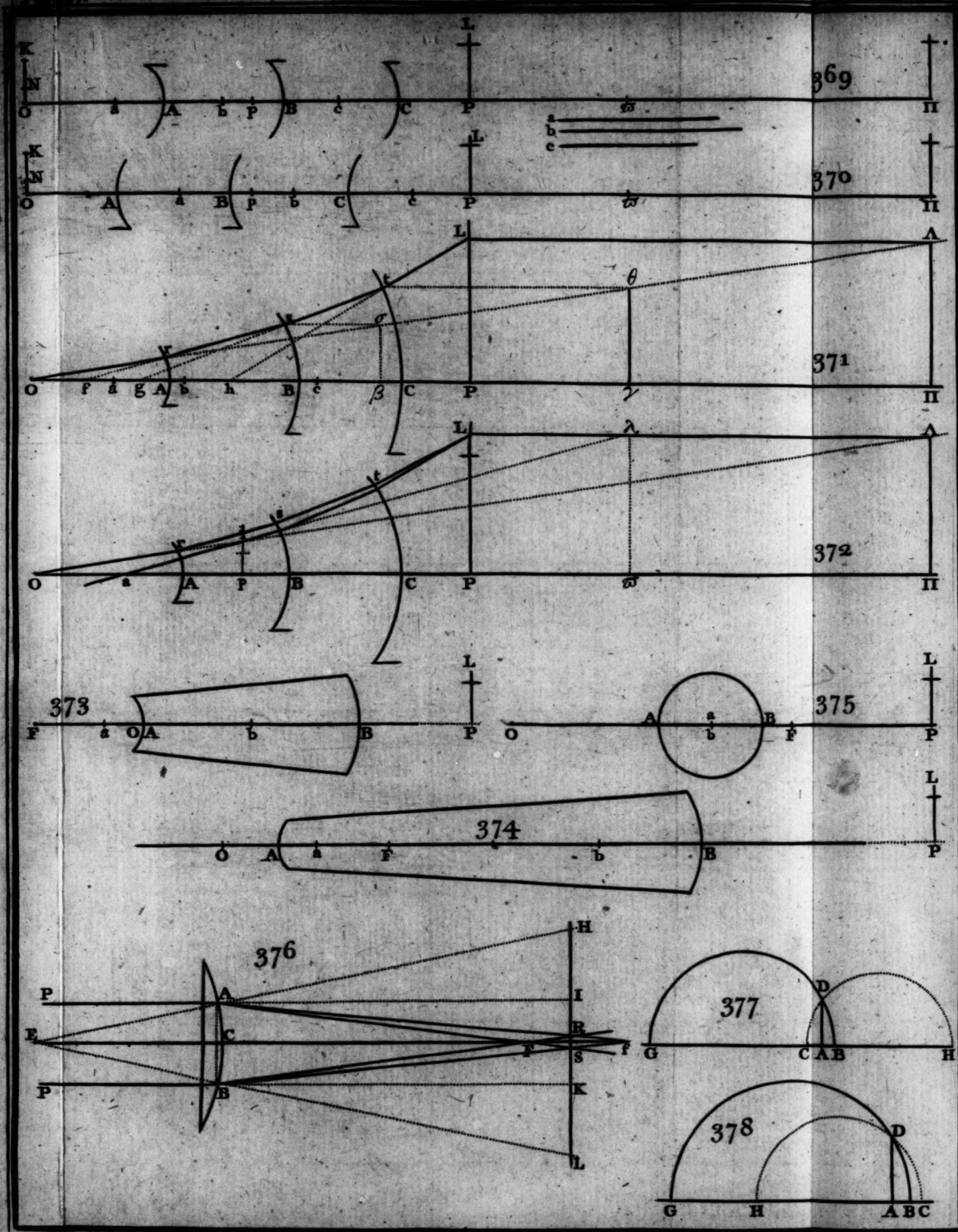
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So far in theory, which is reduced to practice by making the focal distance and the breadth of the surface B very small, that it may not intercept too many rays, in their passage from the object to the large concave C; and by making a moderate hole in the middle of this concave at C for the rays to pass through after reflection from B; and by substituting a convex lens in the place of the concave surface at A, so that the spectator may look towards the object. For supposing this lens to have the same focal distance as the concave surface at A, the apparent distance, magnitude, situation, distinctness and brightness of the object will continue the same as before [a]. a Art. 293.

303. *Corol.* 13. Hence if the spherical surface at B be changed into a plane by increasing its semidiameter and focal distance bB to infinity, the ratio of bB to the interval bc will become a ratio of equality; and then the apparent magnitude will be to the true as c to a by the foregoing corollary. Now when this ratio of Cc to Aa is very great, the common focus a can come but a very little way within the concave Ct, though the eye-glass were placed in the hole at C; therefore Ba and Bc being now equal, or Bc being half of ca must be near half of cC; and consequently the breadth of the plane Bs must be near half the breadth of the concave Ct, to receive the whole pencil reflected from Ct; and then it would also intercept near half the incident rays that come from the object. But if the plane Bs be turned obliquely, to reflect the rays sideways to the eye-glass A, its distance from c and consequently its breadth may be diminished at pleasure without altering the apparent magnitude; which the plane neither increases nor diminishes. For putting b infinite in cor. 9. CP or OP is to $O\Pi$ as c to $-a$. Fig. 366. Fig. 367.

304. *Corol.* 14. When the rays that flow from a near object PL, go in parallel lines to the eye after two reflections at the concave surfaces B and A, or after one reflection at B and a refraction through a convex lens at A whose focal distance is $-a$, the apparent distance of the object from the eye at any point O is $a \times \overline{1 - \frac{BP^*}{b}}$ or $\frac{a}{b} \times \overline{b - BP}$ or $-\frac{a}{b}\, Pb$; which being negative shews that the object appears inverted [b]. Hence the apparent magnitude is to the true, or the true distance to the apparent, as OP to $\frac{a}{b}\, Pb$, or in a ratio compounded of b to a and of OP to Pb. In these reflecting microscopes the object PL being very small can intercept but very few rays in their passage from B to A. Fig. 368. * Art. 299. b Art. 284.

305. *Corol.* 15. Hence again when the object is remote as in a reflecting telescope, consisting of one large concave and a convex eye-glass, the apparent magnitude of the object is to the true as b to a, because the ratio of OP to Pb becomes a ratio of equality, and because the reflecting plane does not alter the apparent magnitude.

306.

306. *Corol.* 16. The place of the eye in reflecting telescopes, where the middle rays of each pencil intersect one another, may be found by
a Art. 239. making Ba, BA, BO continual proportionals [a], because the reflecting surface B corresponds with what I have said of the object-glass of a refracting telescope, in the 279th article as appears by inspection of the 184th figure.

PROPOSITION III.

307. *Having the focal distances and apertures of any number of spherical surfaces, which intercede any given mediums, and are placed at any distances from each other and from the eye and object; it is required to find the apparent distance, magnitude, situation, degree of distinctness and brightness of an object seen through all the mediums: together with the greatest angle of vision and visible area of the object, and the particular aperture which limits them both.*

Ap. distance. Fig. 369, 370. 308. Let PL be an object viewed by the eye at O through any number of spherical surfaces placed at A, B, C, whose centers a, b, c are all in the line OP, and whose focal distances of rays that fall parallel on their sides next the eye are the lines a, b, c. At first let us suppose the semidiameters Aa, Bb, Cc to lye all the same way from their surfaces; and to be all separate from one another and from the eye and object; and that the medium adjoining to the concave side of every surface, is rarer or less refractive than the medium adjoining to its convex side; then take a line

$$O\Pi = OP + \frac{Oa, AP}{a} + \frac{Ob, BP}{b} + \frac{Oc, CP}{c} + \frac{Oa, Ab, BP}{ab} + \frac{Oa, Ac, CP}{ac} + \frac{Ob, Bc, CP}{bc} + \frac{Oa, Ab, Bc, CP}{abc},$$

and it will be the apparent distance of the object.

In all other cases the lines OP, AP, BP, CP must still remain affirmative in this value of $O\Pi$, but any one of the lines Oa, Ob, Oc will be negative if it lyes behind the eye: secondly any one of the lines Ab, Ac, Bc, will be negative if it tends towards the eye from the surface that terminates it: thirdly any one of the focal distances will be negative, if the order of the densities of the mediums remaining as in the first case, the semidiameter of the surface to which it belongs be situated on the other side of its surface; or if, the position of the semidiameter remaining as at first, the densities of the adjoining mediums be transposed to contrary sides of that surface. The sign of every line involved in the foregoing value of $O\Pi$ being thus determined, the sign of every term of it which involves any odd number of negative lines must be looked upon as negative, otherwise as affirmative; then will $O\Pi$, or the sum of all the terms according to their signs, be the apparent distance of the object.

309. And the apparent magnitude of the object will be to its true magnitude as OP to $O\Pi$. Ap. magnitude.

310. And if the value of $O\Pi$ be affirmative the object will appear upright, otherwise inverted. Ap. situation.

311. When $O\Pi$ is affirmative place it before the eye, otherwise behind it. Then imagine the eye to be removed from O to a, that its distance from the center of the next surface may vanish; and here let $a\varpi$ be the apparent distance of the object, to be found and placed by the same rules that $O\Pi$ was; then let ap be to $a\varpi$ as aO to the difference of $O\Pi$ and $a\varpi$, if they lye the same way from O and A, otherwise to their sum; and let the order of the points a, p, ϖ be the same as the order of the points O, p, Π; and from the situation of this point p a judgment may be formed of the degree of distinctness with which the object will appear. Because p is the place of the last image of the object. Ap. distinctness.

312, The determinations of the visual angle and visible area and of the aperture which limits them, are the same as in the proposition for lenses [a]. Visual angle and vis. area. a Art. 252.

313. Since the magnitude of the pupil is subject to be varied by various degrees of light, let NO be its semidiameter, when the object PL is viewed by the naked eye from the distance OP; and upon a plane that touches the eye at O, let OK be the semidiameter of the greatest area visible to another eye at P through all the apertures, to be found as PL was; or which is the same thing, let OK be the semidiameter of the greatest area inlightened by a pencil of rays flowing from P through all the apertures; and when this area is not less than the area of the pupil, the apparent brightness of the point P seen by the refracted rays, will be to its apparent brightness seen through an uniform medium by unrefracted rays, in a ratio compounded of the apparent magnitude, to the true magnitude of any surface at O seen from P, and of the true magnitude, to the apparent, of any surface at P seen from O: but if the inlightened area at O be less than the area of the pupil, the apparent brightness of P seen by the refracted rays, will be to its apparent brightness seen by the unrefracted rays, in a ratio compounded of the two former ratios, and of this inlightened area, to the area of the pupil. This would be the proportion of the apparent brightness if all the rays were transmitted through the mediums, or if only an insensible part of them was stopt by reflections at the surfaces and by the opacity of the matter. Ap. brightness.

DEMONSTRATION.

314. The first part of the demonstration of the first of these propositions [b] gives $O\Pi = OA \times \frac{fA + AB}{fA} \times \frac{gB + BC}{gB} \times \frac{hC + CP}{hC}$. By which theorem $O\Pi$ will be given so soon as fA, gB, hC can be found. These may be Ap. distance. Fig. 371. b Art. 256.

found by the 238th article. For supposing any line AB to be the focal distance of the surface A when the rays fall parallel on its side next the object, we have OB to Oa as OA to Of, and disjointly OB to aB as OA to fA; therefore since we supposed the line a to be the other focal distance of this surface and consequently to be equal to aB[a], we have $fA = \frac{OA,\, a}{Oa -\!+ a}$;

a Art. 224.

$gB = \frac{\overline{fA -\!+ AB} \times b}{fA -\!+ Ab -\!+ b}$; $bC = \frac{\overline{gB -\!+ BC} \times c}{gB -\!+ Bc -\!+ c}$. Whence it is easy to collect, that if the eye at O views an object at B through one surface at A, its apparent distance $O\beta = OB -\!+ \frac{Oa,\, AB}{a}$; that if the eye at O views an object at C through two surfaces at A, B, its apparent distance $O\gamma = OC -\!+ \frac{Oa,\, AC}{a} -\!+ \frac{Ob,\, BC}{b} -\!+ \frac{Oa,\, Ab,\, BC}{ab}$; that if the eye at O views an object PL through three surfaces at A, B, C, its apparent distance $O\Pi = OP -\!+ \frac{Oa,\, AP}{a} -\!+ \frac{Ob,\, BP}{b} -\!+ \frac{Oc,\, CP}{c} -\!+ \frac{Oa,\, Ab,\, BP}{ab} -\!+ \frac{Oa,\, Ac,\, CP}{ac} -\!+ \frac{Ob,\, Bc,\, CP}{bc} -\!+ \frac{Oa,\, Ab,\, Bc,\, CP}{abc}$.
The rules given for determining the signs of every line in this theorem in all other cases, will be evident by observing that the points O, A, B, C, P are fixt in order and position in all cases; that the figures of the surfaces and the position of their centers may be changed from those in the first case to such as are proposed in any other, by increasing their semidiameters Aa, Bb, Cc till they become infinite, and then negative if need be; that any of their focal distances will become infinite and then negative, either when the semidiameter of its surface becomes infinite and then negative, or else when the density of one of the contiguous mediums is gradually altered till it equals the other, and still more till it differs from the other the contrary way; and lastly that during these gradual alterations any line will become negative after it has been nothing or infinite any odd number of times, in passing from its state in the first case to the state proposed in any other.

The determination of the apparent magnitude is evident by art. 141; and that of the apparent situation by the latter part of art. 139.

Ap. distinctness.
Fig. 372.

315. Compleat the rectangle $LP\varpi\lambda$ and join $a\lambda$ meeting $O\Lambda$ in l, and the line lp drawn perpendicular to the axis of the surfaces will be the last image of the object LP. Because the same point L is seen by a ray which falls upon the eye at O in the direction ΛlO, and also by a ray which falls upon the eye at a in the direction λla; and therefore the point l, where the lines $O\Lambda$, $a\lambda$ cross one another, is the focus of the emergent rays. Now since the triangles apl, $a\varpi\lambda$, and also Opl, $O\Pi\Lambda$ are equiangular, we have ap to $a\varpi$ as (pl to $\varpi\lambda$ or $\Pi\Lambda$, or as) Op to $O\Pi$, or as

Op

$Op \mp ap$ or Oa to $O\Pi \mp a\varpi$, according as p falls without or within the line Oa, and consequently according as $O\Pi$ and $a\varpi$ lye the same way or contrary ways from O and a. And the order of the points a, p, ϖ is the same as the order of the points a, l, λ, or of the points O, l, Λ or of the points O, p, Π.

316. First if OK be not less than ON, the area of the pupil will be totally inlightened by the pencil that flows from P. Let $PtsrN$ be a ray of that pencil cutting the surface Ct in t; and supposing the refracting surfaces were all removed, let an unrefracted ray PMN cut the line Ct in M. Then the quantity of refracted light which falls upon the line NO, is to the quantity of unrefracted light which would fall upon it, as the angle CPt to the angle CPM, that is as the apparent magnitude of the line NO, to its true magnitude seen by an eye from P. And therefore by turning the figure round about the axis OP, the quantity of refracted light which fills the pupil, is to the quantity of unrefracted light which would fill it, as the apparent magnitude of any surface at O, to the true. Therefore since the real brightness of any portion or physical point of the retina, is directly as the quantity of light which falls upon it, and inversely as its own magnitude; the apparent brightness of the point P seen by the refracted rays, is to its apparent brightness seen by the unrefracted rays in a ratio compounded of the apparent magnitude, to the true magnitude, of any surface at O seen from P, and of the true magnitude, to the apparent, of any surface at P seen from O. Now if OK be less than ON, then by supposing a lesser pupil whose semidiameter is OK, we have shewn that the apparent brightness of P seen by refracted rays passing through this smaller pupil, (which is the same as if they passed through the larger,) is to its apparent brightness seen by unrefracted rays passing through it, in the given ratio compounded of those abovementioned; and this latter apparent brightness of P seen by unrefracted rays passing through the smaller pupil, is to its apparent brightness seen also by unrefracted rays passing through the larger, in the ratio of the smaller pupil to the larger; which ratio being compounded with the former gives the ratio required. Ap. brightness. Fig. 347, 348.

317. *Corol.* 1. When the object is so remote that the distances OP, AP, BP, CP may be considered as equal to one another, then the apparent distance $O\Pi = OP$ into $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Ob, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$. Fig. 369, 370.

318. *Corol.* 2. Hence when O and b are conjugate focuses of a pencil of rays refracted through any number of surfaces A, B, C, the ratio of Fig. 371.

 the

the angles AOr, Cht, made by the incident and emergent parts of any ray with the axis of the surfaces, is the same as of 1 to $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Ob, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$. For this latter ratio is the same as the ratio of OP to $O\Pi$, by the first corol., that is of the apparent magnitude of a remote object seen from O, to its true magnitude seen from O or b by the naked eye, or as the angle at O to the angle at b.

319. *Corol.* 3. Hence if O be the focus of incident rays, the focus b of the emergent rays from the last surface C, may be found by taking Cb to $O\gamma$ or $OC + \frac{Oa, AC}{a} + \frac{Ob, BC}{b} + \frac{Oa, Ab, BC}{ab}$ as 1 to $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Oa, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$; and by placing Cb contrary to the course of the rays if the second and third terms of this proportion have like signs, otherwise according to their course. For compleating the rectangle $\gamma Ct\theta$, Cb is to $O\gamma$ as the angle $\gamma O\theta$ to the angle Cbt[a], that is in the ratio abovementioned, by cor. 2.

a Art. 222.

320. *Corol.* 4. When the distances of the surfaces and object from one another, and their focal distances a, b, c are such, that $1 + \frac{AP}{a} + \frac{BP}{b} + \frac{CP}{c} + \frac{Ab, BP}{ab} + \frac{Ac, CP}{ac} + \frac{Bc, CP}{bc} + \frac{Ab, Bc, CP}{abc} = o$, the rays of any pencil will fall upon the eye in parallel lines; and then the apparent distance $O\Pi$ will be equal to $a\varpi = aP + \frac{ab, BP}{b} + \frac{ac, CP}{c} + \frac{ab, Bc, CP}{bc}$; and this apparent distance being invariable, the apparent magnitude, situation and degree of distinctness and brightness will also be invariable wherever the eye is placed. For the rays will fall parallel upon the eye when the lines $O\Lambda$, $a\lambda$ are parallel, and consequently when $O\Pi = a\varpi$ or $O\Pi - a\varpi = o$; and the value of $a\varpi$ is found by making Oa vanish in the general value of $O\Pi$[b].

b Art. 308.

321. *Corol.* 5. Hence when the object is so remote that the distances AP, BP, CP, may be considered as equal to one another, the rays which come parallel upon the first surface will emerge parallel from the last, if the surfaces be so disposed that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{Ab}{ab} + \frac{Ac}{ac} + \frac{Bc}{bc} + \frac{Ab, Bc}{abc} = o$; and on the contrary: and then $O\Pi$ (or $a\varpi$) $= OP$ into $1 + \frac{ab}{b} + \frac{ac}{c} + \frac{ab, Bc}{bc}$. For example let AB be the axis of a solid medium, as of glass, and F be the common focus of rays that come parallel upon one of its surfaces and go parallel from the other; and when the surface A is concave and B convex, by changing the sign of b (because the order of the densities of the

Fig. 373.

the adjoining mediums is changed) the former of these equations gives $\frac{1}{a} - \frac{1}{b} - \frac{Ab}{ab} = 0$; whence $b = a + Ab = FA + Ab = Fb$; and the latter equation gives $OP : O\Pi :: b : b - ab :: Fb : Fb - ab$ or $Fa :: bB : aA$* [* Art. 224.] $:: FB : FA$. And when the surface A is convex the same proportion comes out by changing the sign of a. [Fig. 374.] Thus telescopes might be made of one continued solid; but to magnify much they must be made pretty long, and consequently but few rays could be transmitted through so great a length of glass; as is found by experience.

322. *Corol.* 6. Let aF be the focal distance of a sphere whose center is a or b and diameter AB, and the object PL will appear through it at the distance $O\Pi = OP - \frac{OaP}{aF}$ if the sphere be denser than the ambient medium, or at the distance $O\Pi = OP + \frac{OaP}{aF}$ if it be rarer: and these expressions are the same as in a lens of no thickness[a]. [Fig. 375.] For in the general value of $O\Pi = OP + \frac{Oa, AP}{a} + \frac{Ob, BP}{b} + \frac{Oa, Ab, BP}{ab}$; instead of a, b, Ob, Ab, put $-a$, $-b$, Oa, Aa, respectively[b]; and in the denser sphere we have $O\Pi = OP - \frac{Oa, AP}{a} - \frac{Oa, BP}{b} + \frac{Oa, Aa, BP}{ab} = OP - Oa \times \overline{\frac{AP}{a} + \frac{BP}{b} - \frac{Aa, BP}{ab}} = OP - Oa \times \overline{\frac{AP}{a} + \frac{BP}{a} \times \frac{a - Aa}{b}} = OP - Oa \times \frac{AP + BP}{a} = OP - Oa \times \frac{2aF}{a} = OP - \frac{OaP}{aF}$ because $aF = \frac{1}{2}a$, as is easy to collect from the 227th article or from cor. 3. And in the rarer sphere the sign of aF must be changed.

[a] Art. 248.

[b] Art. 308.

323. *Corol.* 7. Let an object PL be viewed by the eye at O through any number of given mediums distinguished from one another by parallel planes Ar, Bs, Ct; and when a ray goes from the eye to the object let the sine of incidence be to the sine of refraction at the plane Ar as q to r, at the plane Bs as r to s, at the plane Ct as s to t; then take a line $O\Pi = OA + \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; and the object will appear at the same distance, of the same magnitude, in the same situation and with the same degree of distinctness and brightness through all the mediums as it would do when viewed in one uniform medium from the distance $O\Pi$. [Fig. 344.]

This may be gathered from the general theorem for $O\Pi$, when varied a little by introducing the semidiameters of the surfaces instead of their focal distances, whose relation is given by the 224th article, and then by making the semidiameters infinite; but more readily as follows. By art. 223. we have $fA : OA :: q : r$; $gB : fA + AB :: r : s$; $hC : gB + BC :: s : t$. Hence $fA = \frac{q}{r} OA$; $gB = \frac{r}{s} \times \overline{fA + AB} = \frac{q}{s} OA + \frac{r}{s} AB$;

AB; $bC = \frac{s}{t} \times \overline{gB + BC} = \frac{q}{t} OA + \frac{r}{t} AB + \frac{s}{t} BC$. But $O\Pi = OA \times \frac{fA + AB}{fA} \times \frac{gB + BC}{gB} \times \frac{bC + CP}{bC} = OA \times \overline{1 + \frac{AB}{fA}} \times \overline{1 + \frac{BC}{gB}} \times \overline{1 + \frac{CP}{bC}}$ as before[a]. Whence it is easy to collect, that if the eye at O views an object at B through one plane Ar, its apparent distance $O\beta = OA + \frac{r}{q} AB$; that if the eye at O views an object at C through two planes Ar, Bs, its apparent distance $O\gamma = OA + \frac{r}{q} AB + \frac{s}{q} BC$; that if the eye at O views an object at P through three planes Ar, Bs, Ct, its apparent distance $O\Pi = OA + \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; and so on. To find the place of the last image, put $OA = o$, then $A\varpi = \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; by which it appears that ΠA and $\varpi\lambda$, and consequently the last image pl do all coincide in place and magnitude; which makes this corollary very evident, supposing as to the brightness that no rays are reflected from the surfaces; and that all the mediums are equally transparent.

Fig. 345.

a Art. 314.

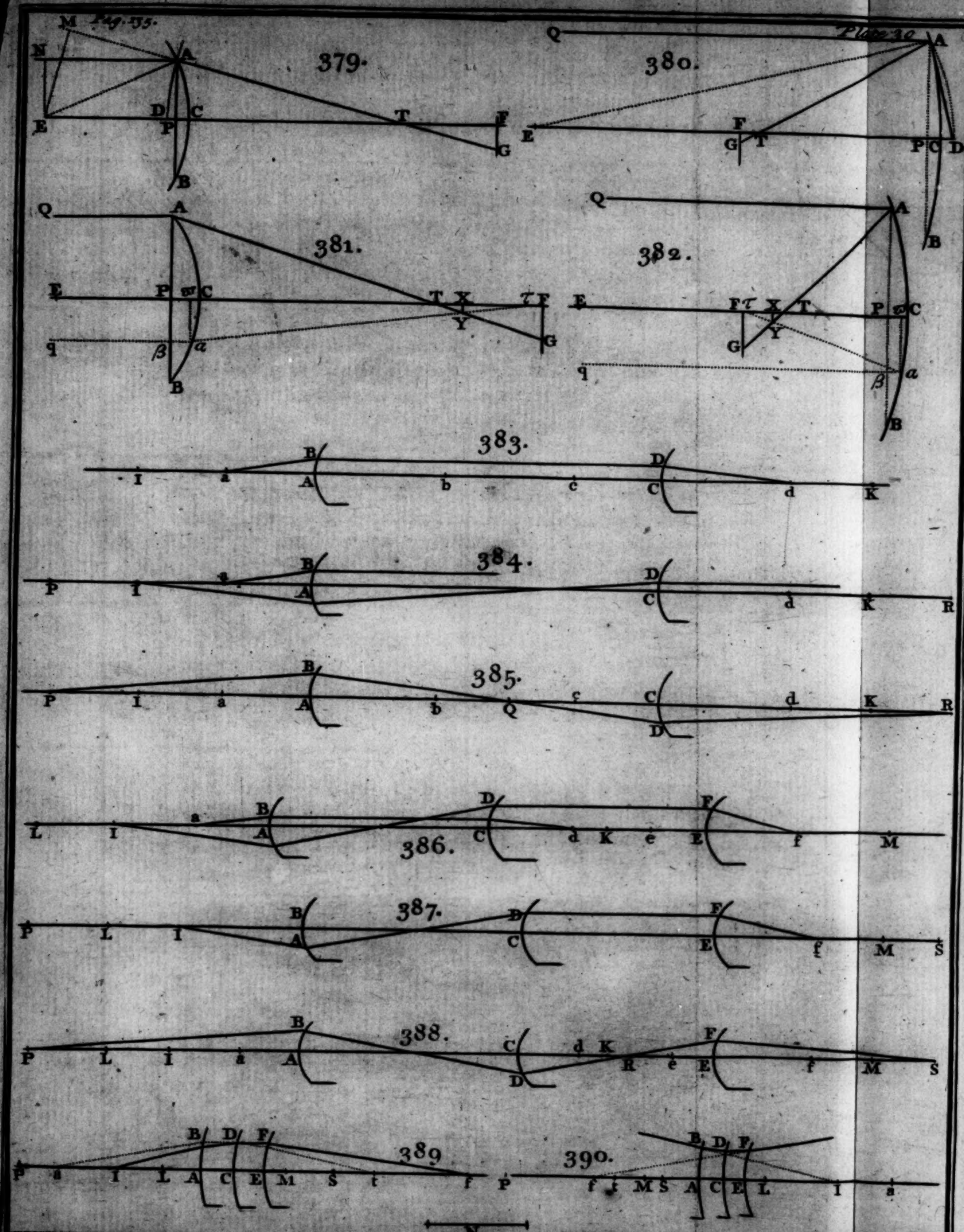
CHAPTER VI.

To determine the aberrations of rays, from the geometrical focus, caused by their unequal refrangibility, and also by the sphericalness of the figure of reflecting and refracting surfaces.

PROPOSITION I.

324. LET *the common sine of incidence be to the sine of refraction of the least refrangible rays as* I *to* R, *and to the sine of refraction of the most refrangible rays as* I *to* S; *and the diameter of the least circular space into which heterogeneal parallel rays can be collected, by a spherical surface or by a plano-convex lens, will be to the diameter of its aperture, in the constant ratio of* S − R *to* S + R − 2 I.

Fig. 376. For let an heterogeneal ray PA fall upon a spherical surface ACB, and let it be separated by refraction into the rays AF, Af, cutting the axis EC, drawn parallel to PA, in F and f. Take the arch CB equal to CA and let another heterogeneal ray PB, coming parallel to PA be refracted into the lines BF, Bf, cutting the two former rays in R and S. Join RS and produce it till it meets the incident rays produced in I and K, and the perpendiculars EA, EB to the refracting surface at the points A, B, in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in H and L. And when AB, the breadth of the aperture or of the pencil, is but moderate, and consequently the refractions at A, B but small, the angles of incidence and refraction, HAI, HAR, HAS, or the arches that measure them, or their perpendicular subtenses HI, HR, HS, will be to each other very nearly in the same given ratios as those of the sines I, R, S of those angles[a]. And disjointly the differences of those subtenses will be proportionable to the differences of these sines: that is, the line $RS : RI :: S-R : R-I$, and doubling the consequents, $RS : 2RI$ or $IK - RS :: S-R : 2R-2I$; and conjointly $RS : IK$ or $AB :: S-R : S+R-2I$. From this given ratio of RS to AB in which they increase or decrease together, it appears that all the intermediate rays which fall upon AB will pass through RS. And when parallel rays fall perpendicularly upon the plane side of a plano-convex lens, they are refracted only at their emergence from its convex surface; and so the aberrations are the same in both cases. *Q. E. D.* a Art. 204. 220.

325. *Corol.* 1. Hence the diameter RS, of the circle of aberrations that contains all the incident rays, is a 55th part of the diameter AB of the aperture of a plano-convex glass, whatever be its focal distance. For supposing AR and AS to be the outmost red and indigo rays, their sines of incidence and refractions I, R, S are to each other as 50, 77, 78[b]. Whence $S-R$ is to $S+R-2I$ as 1 to 55. b Art. 179.

326. *Corol.* 2. The diameter of the least circle that can receive the rays of any single colour or of several contiguous colours is also determinable from the proportions of their sines. Thus all the orange and yellow is contained in a circle whose breadth is the 260th part of the breadth of the aperture of the plano-convex glass; the sines of the outermost orange AR and yellow AS being to the common sine of incidence as $77\frac{1}{8}$ and $77\frac{1}{3}$ to 50[c]. c Art. 179.

327. *Corol.* 3. In different surfaces, or plano-convex glasses, the angles of aberration RAS are as the breadths of the apertures AB directly and as the focal distances CF inversely; because any angle, as RAS, is as its subtense RS directly and as its radius AR or CF inversely.

LEMMA.

328. The versed sines AB, AC of very small arches BD, CD, of unequal circles BDG, CDH, that have the same right sine AD, are reciprocally proportionable to their diameters BG, CH very nearly; that is, $AB : AC :: CH : BG$. Fig. 377, 378.

For since the rectangles under BAG and CAH are each equal to the square of AD[d], and consequently to each other; their sides are reciprocally proportionable[e], that is AB is to AC as AH to AG or as CH to BG very nearly, when the versed sines are incomparably less than the diameters themselves[f]. *Q. E. D.* d Euc. VI. 13. e Euc. VI. 14. f Art. 204.

PROPO-

PROPOSITION II.

Fig. 379. 329. *When homogeneal parallel rays* NA, EC *fall upon a spherical surface* AC *whose center is* E, *the longitudinal aberration* FT *of any refracted ray* AT *from* F *the focus of the pencil, is to the versed sine of the arch* AC *intercepted between the point of incidence and the axis* ECF, *in the given ratio of the square of the sine of refraction, to the rectangle under the sine of incidence and the difference of the sines very nearly; and the aberration is the same when the rays fall perpendicularly upon the plane side of a plano-convex lens.*

For when the refraction is made in the passage of a ray NA from a denser to a rarer medium, it appears by the description of a caustick in the 74th article, that the intersection T of the refracted ray AT with the axis ECF, lyes between the refracting surface and its focus F. With the center T and semidiameter TA having described the arch AD cutting the axis in D, draw the sine AP of the arches AC, AD, and also EN and EM the sines of incidence and refraction, for which put n and m; then because the triangles ETM, ATP are similar, it will be as $ET:TA$ or $TD::(EM:AP$ or $EN::)\,EF:FC$*; and disjointly $TF:EF::(FC-TD$ or$)\,TF-CD:FC$; and alternately $TF:TF-CD::EF:FC$; and disjointly $TF:CD::(EF:EC::)\,m:m-n$ [a]. Again since $(PD:PC::CE:DT$ [b] or FC [c], and conjointly$)\,CD:CP::(EF:FC::)\,m:n$; by compounding this and the foregoing proportion, it will be as $TF:CP::mm:\overline{m-n},n$. Q.E.D.

* Art. 225.
a Art. 224.
b Art. 328.
c Art. 204.

330. *Corol.* 1. The segment $ACBPA$ may be considered as a plano-convex lens; and when rays fall parallel upon its plane side, the longitudinal aberration of the extream ray falling upon A is equal to $\frac{9}{2}$ of its thickness PC, as appears by putting 3 and 2 for m and n respectively.

331. *Corol.* 2. Also this aberration $FT=\frac{mm}{m-n,n}\times\frac{AP^2}{2EC}=\frac{mm}{\overline{m-n}|^2}\times\frac{AP^2}{2CF}$. For $PC=\frac{AP^2}{2EC}$ very nearly [d], and $EC=\frac{m-n}{n}\times CF$ [e].

d Euc. VI. 13.
e Art. 224.

332. *Corol.* 3. Let the refracted ray ATG produced, cut the line FG, perpendicular to the axis, in G, and the lateral aberration $FG=\frac{mm}{nn}\times\frac{AP^3}{2EC^2}=\frac{mm}{\overline{m-n}|^2}\times\frac{AP^3}{2CF^2}$. For $FG:TF::AP:TP$ or CF or $\frac{n}{m-n}\times CE$.

333. *Corol.* 4. When the semidiameter of the convexity or the focal distance is given, the longitudinal aberrations are as the squares, and the lateral aberrations as the cubes, of the linear apertures of a plano-convex lens.

PROPO-

PROPOSITION III.

334. *When parallel rays* QA, EC *are reflected from a spherical concave* ACB *whose center is* E *and whose aperture* ACB *is but small, the longitudinal aberration* TF *of the extream ray* AT *from the geometrical focus* F, *is equal to half the versed sine* CP *of the semiaperture* AC *very nearly.* Fig. 380.

In fig. 379. imagine EM, the sine of refraction to be diminished to nothing and then to become negative and equal to EN the sine of incidence, and the refraction of the ray to be changed to reflection as in fig. 380; and by the former proposition it will be as $TF : CP :: mm : \overline{-m-n}, n :: nn : -2nn :: 1 : -2$.

But the particular proof is this. By the last lemma the versed sine CP nearly equals $\frac{1}{2}$ the versed sine PD of the arch AD whose center is T and semidiameter TA or TE or $\frac{1}{2}$ the semidiameter of the arch AC[a] very nearly. But $2TF = 2TE - 2EF = ED - EC = CD$ exactly or CP nearly. Therefore $TF = \frac{1}{2} CP$ nearly. Fig. 380. a Art. 205.

335. *Corol.* 1. We had $2TF = CD$ exactly; which is the excess of the secant ED of the arch AC above its radius EA. For joyning AD the angle DAE in the semicircle DAE is a right one.

336. *Corol.* 2. The longitudinal aberration $TF = \frac{AP^2}{4CE}$. For $CP = \frac{AP^2}{2CE}$ nearly[b]. b Euc. VI. 13.

337. *Corol.* 3. The lateral aberration $FG = \frac{AP^3}{2CE^2}$. For $FG : FT :: AP : PT$ or $\frac{1}{2} CE$ nearly.

338. *Corol.* 4. When the diameter of the concave, or its focal distance, is given, the longitudinal aberrations are as the squares and the lateral ones as the cubes of the diameters of the apertures.

PROPOSITION IV.

339. *When parallel rays of any one sort are refracted by a plano-convex object-glass, or when rays of all sorts are reflected by a spherical concave, the diameter of each circle of aberrations caused by the sphericalness of the figures, is equal to $\frac{1}{2}$ the lateral aberration of the extream ray in each; and therefore is given by the former propositions.*

Let $\alpha Y\tau$ be any refracted or reflected ray cutting the axis ECT in τ, and the extream ray ATG, that comes from the contrary side of the axis, in Y. Draw YX perpendicular to the axis, and supposing the line ATG immoveable, as the point of incidence α moves from the vertex C, the perpendicular XY will first increase, because the angle $C\tau\alpha$ continually increases, Fig. 381, 382.

creases, and afterwards will decrease, because the line $T\tau$ continually decreases; and when XY is the greatest, it is evident that all the rays, incident upon the same side of the axis as it self, will pass through it. To find its greatest quantity, let the incident ray $q\alpha$ cut the chord APB in β, and supposing the variable aperture $P\beta = v$, the variable $TX = x$ and the given lines $PA = a$, $PT = f$, $TF = b$; by corr. 4. prop. 2 and 3, the aberration $F\tau$ is to the aberration FT (b) as $\varpi\alpha^2$ or $P\beta^2$ (vv) to PA^2 (aa). Wherefore $F\tau = \frac{vv}{aa} b$ and thence $TF - F\tau = T\tau = \frac{b}{aa} \times \overline{aa - vv}$. Again $PT\,(f) : PA\,(a) :: TX\,(x) : XY = \frac{ax}{f}$; also $\varpi\alpha\,(v) : \varpi\tau$ or $PT\,(f) :: XY\left(\frac{ax}{f}\right) : X\tau = \frac{ax}{v}$. Hence again $T\tau$, or $X\tau + XT = \frac{ax}{v} + x = \frac{b}{aa} \times \overline{aa - vv}$ found before; or $\frac{x}{v} \times \overline{a + v} = \frac{b}{aa} \times \overline{a + v} \times \overline{a - v}$. Whence $x = \frac{b}{aa} v \times \overline{a - v}$, and therefore x or TX is the greatest possible when the rectangle $v \times \overline{a - v}$, or $P\beta \times \beta B$ is greatest, that is when its sides $P\beta$,
a Euc. VI. 13. βB are equal[a], or when $v = \frac{1}{2} a$. Substitute this value for v in the last equation and it gives the greatest value of $x = \frac{1}{4} b$ or the greatest $TX = \frac{1}{4} TF$, and therefore the greatest $XY = \frac{1}{4} FG$, because $TX : XY :: TF : FG$, and this XY turned about the axis PX describes the circle of aberrations through which all the rays falling upon AB will just pass. *Q.E.D.*

PROPOSITION V.

340. *The circle of aberrations caused by the sphericalness of the figure of the object-glass of a telescope, compared with the circle of aberrations caused by the unequal refrangibility of rays, is altogether inconsiderable.*

Newt. Opt. p. 83. For if the object-glass be plano-convex and the plane side be turned towards the object, and the diameter of a sphere whereof this glass is a segment be called D, and the semidiameter of the aperture of the glass be called S, and the sine of incidence out of glass into air be to the sine of refraction as n to m; the rays which come parallel to the axis of the glass shall in the place where the image of the object is most distinctly made, be scattered all over a little circle whose diameter is $\frac{mm}{nn} \times \frac{S^3}{DD}$ very nearly, if they were all equally refrangible by article 339 and 332. As for instance if the sine of incidence n be to the sine of refraction m as 20 to 31, and if D, the diameter of the sphere to which the convex side of the glass is ground, be 100 foot or 1200 inches, and consequently the telescope about 100 foot long[b], and S the semidiameter of the aperture be 2
b Art. 224. inches; the diameter of this circle of aberrations, that is $\frac{mm}{nn} \times \frac{S^3}{DD}$, will

be

be $\frac{31 \times 31 \times 8}{20 \times 20 \times 1200 \times 1200}$ or $\frac{961}{72000000}$ parts of an inch. But the diameter of the little circle through which these rays are scattered by unequal refrangibility, will be about the 55th part of the breadth of the aperture of the object-glass[a], which is here 4 inches. And therefore the aberration arising from the spherical figure of the glass, is to the aberration arising from the different refrangibility, as $\frac{961}{72000000}$ to $\frac{4}{55}$, that is as 1 to 5449; and therefore being in comparison so very little, deserves not to be considered in the theory of telescopes. If we suppose the little circle of aberrations arising from unequal refrangibility, to be 250 times narrower than the circular aperture of the object-glass, it would contain all the orange and yellow, and would permit the other fainter and darker colours to pass by it[b], which perhaps may scarce affect the sense; yet even in this case the aberration caused by the spherical figure, would be to the aberration caused by the unequal refrangibility, in a 100 foot telescope, but as $\frac{961}{72000000}$ to $\frac{4}{250}$ or only as 1 to 1200, which sufficiently proves the proposition. *Q. E. D.*

a Art. 325.

b Art. 326. Newt. Opt. p. 88.

341. *Corol.* 1. If the focal distances and apertures of a reflecting concave and a plano-convex glass be both the same, the diameter of the circle of aberrations, caused by their figures, will be above 30 times less in the reflecter than in the refracter. For these diameters are $\frac{AP^3}{16CF^2}$ and $\frac{mm}{\overline{m-n}|^2} \times \frac{AP^3}{4CF^2}$ by art. 339, 337 and 332; which are as $\frac{1}{4}$ to $\frac{mm}{\overline{m-n}|^2}$ or $\frac{31 \times 31}{11 \times 11}$. Hence if the length of each telescope be 100 foot, the lateral aberrations in the reflecter would be 30×5449 or 163470 times less than the lateral aberrations caused by unequal refrangibility in the refracter.

342. *Corol.* 2. The number of pencils, some of whose rays are mixed together in every point of a confused picture, is as the area of the circle of aberrations of the rays in any one pencil; and consequently the mixture of the rays of different pencils, caused by the sphericalness of the figure of an object-glass, if they were all alike refrangible, would be to their mixture caused by their unequal refrangibility, as 1 to 5449×5449 or 29691601 in the present instance. For conceiving any point in the confused picture to be a center of a circle of aberrations, it is manifest that all other equal circles of aberrations, whose centers fall upon the first mentioned circle will cover its center; that is some rays of as many pencils will be mixed in this center as there are points in the circle it self; or, which is the same thing, the number of pencils mixed in this center is as the area of the circle of aberrations.

Chapter VII.

A refracting or reflecting telescope being given, whose aperture and eye-glass are adjusted by experience, to determine the length, aperture and eye-glass of another telescope, through which an object shall appear as bright and distinct as in the given one, and magnified as much as shall be required.

Proposition I.

343. In *all sorts of telescopes and double microscopes, the apparent indistinctness of a given object, is as the area of a circle of aberrations in the focus of the object-glass directly, and as the square of the focal distance of the eye-glass inversely.*

For in vision with the naked eye or with glasses, the apparent indistinctness of a given object, is as the area of a circle of aberrations in its picture painted upon the retina. Because any one sensible point of the retina, being the center of a circle of aberrations, will at once be affected by a mixture of the rays of as many distinct pencils, as there are sensible points in the area of that circle [a]; and so will at once convey to the mind a mixt or confused sensation of the same number of visible points in the object, from whence those pencils flowed; and this number of points is as the magnitude of the area of a circle of aberrations, whatever be the magnitude of a sensible point of the retina. Now in vision with telescopes, the diameter of a circle of aberrations in the picture upon the retina, is as the apparent magnitude of the diameter of the corresponding circle of aberrations in the common focus of the glasses [b], that is as the angle subtended by this diameter at the center of the eye-glass [c]; that is as the diameter it self directly, and the focal distance of the eye-glass inversely [d]. And so the area of that circle of aberrations upon the retina, is as the area of the corresponding circle of aberrations in the focus of the object-glass directly, and as the square of the focal distance of the eye-glass inversely [e].

a Art. 342.
b Art. 91.
c Art. 120.
d Art. 222.
e Euc. XII. 2.

344. *Corol.* In all sorts of telescopes and double microscopes a given object appears equally distinct, when the focal distances of the eye-glasses are as the diameters of the circles of aberrations in the focus of the object-glasses.

345. The alteration in the confusion which may arise from aberrations caused by the eye-glasses, is not here regarded, as being inconsiderable. We only consider the confusion of those points in the image which lye very near the axis of the telescope, as of the point q in fig. 181. Now if this point was perfectly distinct the rays going from it would emerge from the

the eye glaſs in parallel lines without ſenſible error; becauſe the breadth of this cylinder of rays is exceeding ſmall compared to the breadth of the eye-glaſs, being in proportion to the breadth of the aperture of the object-glaſs as their focal diſtances; and the refractions at ſo ſmall a diſtance from the axis are ſufficiently true and regular. It is the largeneſs of the aperture of the object-glaſs and of its focal diſtance, which cauſes the irregularity in its refractions. Add to this that the differently refrangible rays cannot be ſeparated ſenſibly in going ſo ſhort a diſtance as between the eye-glaſs and the eye. Beſides this we find by experience that objects and images diſtinct in themſelves, appear ſufficiently diſtinct through very ſmall eye-glaſſes when their apertures are ſmall. This remark will be demonſtrated more diſtinctly in the 11th chapter, where the ſame ſubject is handled more fully.

PROPOSITION II.

346. *In refracting teleſcopes the apparent indiſtinctneſs of a given object, is directly as the area of the aperture of the object-glaſs, and inverſely as the ſquare of the focal diſtance of the eye-glaſs.*

This appears from prop. 1, becauſe the area of the circle of aberrations at the focus of the object-glaſs is as the area of its aperture [a]; and becauſe the aberrations ariſing from the eye-glaſs [b], and from the ſphericalneſs of the figure of them both are inconſiderable [c].

a Art. 324.
b Art. 345.
c Art. 340.

347. *Corol.* In refracting teleſcopes a given object appears equally diſtinct, when the diameters of the apertures of their object-glaſſes, are as the focal diſtances of their eye-glaſſes.

PROPOSITION III.

348. *In all ſorts of teleſcopes and double microſcopes the apparent brightneſs of a given object is as the ſquare of their linear apertures directly and as the ſquare of their linear amplifications inverſely.*

For if the ſquares of the linear amplifications, that is if the areas of the pictures upon the retina were the ſame, their brightneſs would be as the quantities of light coming through the areas of the apertures, that is as the ſquares of the linear apertures; and if the apertures or quantities of light were the ſame, the brightneſs of the pictures would be as their areas inverſely or as the ſquares of the linear amplifications inverſely. Therefore when neither the apertures nor the amplifications are the ſame, the brightneſs is as the ſquare of the linear apertures directly, and as the ſquare of the linear amplifications inverſely. *Q. E. D.*

349. *Corol.* 1. Hence in refracting and reflecting teleſcopes a given object appears equally bright, when their linear apertures are as their linear amplifications, that is as the focal diſtances of the object-glaſſes directly and as the focal diſtances of the eye-glaſſes inverſely.

350.

350. *Corol.* 2. If the breadth of the aperture of a given object-glaſs and the focal diſtance of the eye-glaſs be each increaſed in any given ratio, the a Art. 347. diſtinctneſs will remain the ſame as before [a]; and the linear amplification b Art. 120. will be diminiſhed in the ſame ratio [b]; but the apparent brightneſs will be increaſed in a ratio quadruplicate of the former ratio by this propoſition; and on the contrary.

Dioptr. p. 215. 351. *Hugens* obſerves that the ſame degrees of diſtinctneſs here demonſtrated do not exactly agree with experience, as he found by looking at the ſame object through different teleſcopes, or through the ſame teleſcope with different apertures; and that through the larger aperture the object appeared not quite ſo diſtinct as through the ſmaller. He found alſo that in viewing objects of different brightneſs through the ſame aperture, the apparent indiſtinctneſs of the brighter object was a little grearer than that of the duller: and therefore the aperture adjuſted for the duller planets may be ſomewhat larger than for the brighter.

PROPOSITION IV.

352. *In reflecting teleſcopes the apparent indiſtinctneſs of a given object is as the ſixth power of the diameter of the aperture of the object-metal directly, and as the fourth power of its focal diſtance inverſely, and alſo as the ſquare of the focal diſtance of the eye-glaſs inverſely.*

For the area of a circle of aberations in the focus of the object-metal is as the ſixth power of its linear aperture directly and as the fourth power a Art. 339. 337. of its focal diſtance inverſely [a]; and therefore the apparent indiſtinctneſs of the object, is as the ſixth power of the linear aperture directly, as the fourth power of the focal diſtance of the object-metal inverſely, and as the b Art. 343. ſquare of the focal diſtance of the eye-glaſs inverſely [b]. *Q. E. D.*

353. *Corol.* In reflecting teleſcopes a given object appears equally diſtinct when the cubes of the linear apertures of the object-metals, are as the ſolids whoſe baſes are the ſquares of the focal diſtances of the object-metals, and heights are the focal diſtances of the eye-glaſſes: or when the focal diſtances of the eye-glaſſes are as the cubes of the linear apertures of the object-metals, applyed to the ſquares of their focal diſtances.

PROPOSITION V.

354. *In refracting teleſcopes of various lengths a given object will appear equally bright and equally diſtinct, when their linear apertures and focal diſtances of their eye-glaſſes are ſeverally in a ſubduplicate ratio of their lengths or focal diſtances of their object-glaſſes: and then alſo their linear amplifications will be in a ſubduplicate ratio of their lengths.*

For to ſhew the object equally bright, the rectangle under the linear aperture and the focal diſtance of the eye-glaſs muſt be as the length of the

the telescope [a], and to shew it equally distinct the linear aperture must be as the focal distance of the eye-glass [b]; and therefore to perform both things together, the square of the linear aperture, and also the square of the focal distance of the eye-glass, must be severally (as the rectangle under each, or) as the length of the telescope; and consequently the linear aperture, and also the focal distance of the eye-glass, as the square root of that length. Now the linear amplification was as the linear aperture [c], or by this demonstration, as the square root of the length of the telescope. *Q. E. D.*

a Art. 349.
b Art. 347.
c Art. 347.

355. *Hugens*'s standard telescope 30 foot long, or 360 inches, bears an aperture whose breadth is 3 inches, and an eye-glass whose focal distance is 3 inches and 3 tenths. From whence he has given us the following table of apertures and eye-glasses for other telescopes [d], computed by the following rule.

Diop. p. 210.
d Art. 364.

Multiply the number of feet in the focal distance of any proposed object-glass by 3000, and the square root of the product will give the breadth of its aperture in hundredth parts of an inch. And the same breadth of the aperture, increased by a tenth part of it self, gives the focal distance of the eye-glass in hundredth parts of an inch. And the magnifying powers are as the breadths of the apertures.

For since the standard telescope has 30 foot focal distance of its object-glass, put F for the number of feet in any other focal distance, and say by the proposition as $\sqrt{30}$ to $\sqrt{F}$, so is the standard aperture 3 inches, or 300 centesimals or $\sqrt{300 \times 300}$, to the aperture sought; which therefore is $\sqrt{3000F}$ in centesimals of an inch. The focal distance of the eye-glass of the standard telescope is $3\frac{3}{10}$ inches, that is a tenth part more than the breadth of the aperture of the object-glass; consequently the focal distance of the new eye-glass must be a tenth part more than the linear aperture of the new object-glass, by the last proposition.

356. He also adds the following directions how to suit these telescopes to all sorts of objects seen either by day or by night. They are proportioned in the following table for astronomical observations, and therefore will require more light when used in the day time. For when the eye is dazled with the brightness of the day, objects will appear through them but obscure, which in the night are sufficiently bright. Therefore (says *Hugens*) when I used these telescopes to observe objects by day-light, by experience I found it requisite to change the eye-glasses for others whose focal distances were double the former. By this means the apparent brightness became quadruple, because the surfaces of the images in the bottom of the eye were diminished in the same proportion [e]. For as the aperture remains unaltered, so does the quantity of light, and therefore it illuminates a lesser space so much the more. Now if the aperture was increased

Dioptr. p. 215.
e Art. 120.

with-

without changing the eye-glass, the brightness would be increased too, but then the mist arising from greater aberrations would also be greater; and therefore this remedy must not be used.

357. But one may ask this question, since by substituting an eye-glass of a longer focal distance, the apparent indistinctness hitherto examined is diminished, why may not the aperture of the object-glass be so far increased, till the same degree of indistinctness returns again as belongs to a telescope regulated by the table? For from hence more light is gained

a Art. 350. and the distinctness is not altered [a]. The answer is this, which I hinted

b Art. 351. before [b], that the mist arising from *Newton's* aberration, though the same in quantity, becomes more sensible in proportion to the brightness of the image. For the brightness of the mist increases at the same time. And we find by experience, that as soon as the apertures of those day-light telescopes are increased, the mist arising from the aberrations of a brighter object begins to be troublesome. The apertures therefore must not be altered.

358. Again one may ask, if a telescope fitted for Saturn be applyed to the Moon, which is 100 times brighter (I mean in each equal parts, though not in the whole, as being 10 times nearer to the Sun;) one may ask I say whether the breadth of the aperture and the focal distance of the eye-glass may not both be lessened in the same proportion, to make the regions of the moon no brighter than those of saturn, but much greater in appearance than before. For instance, in a 30 foot telescope, if 3 inches, the breadth of the aperture be reduced to $\sqrt{\frac{9}{10}}$ of an inch, which is somewhat less than ($\sqrt{\frac{9}{9}}$ or) a third part of the former, and also the focal distance of the eye-glass be shortened in the same proportion; the proportion of the apparent brightness in these two telescopes, the object being

c Art. 350. the same, would be quadruplicate of 3 to $\sqrt{\frac{9}{10}}$ [c] that is as 100 to 1; and since the regions in the moon are 100 times brighter in themselves than those in saturn, the moon would appear in the darker telescope just as bright as saturn did in the lighter. But the apparent indistinctness hither-

d Art. 350. to considered would also be the same in both [d], and the amplification of

e Art. 120. the moon would be greater than that of saturn in the ratio of 3 to $\sqrt{\frac{9}{10}}$ [e], which is more than triple. So that this reduction of the aperture and eye-glass seems very advantageous; but in reality it is quite otherwise; and that for two reasons. First because the minute parts of the moon may be better discerned when all the light remains in the telescope, than when it is reduced to an 100th part, though not in the same proportion. The other reason is that when the aperture is too much contracted, the out-lines that circumscribe the pictures in the eye become confused; which is carefully to be minded, and also what are the limits of this confusion. This is certain that as the aperture is contracted, the slender pencils or cylinders of rays, that emerge from the eye-glass into the eye, are also contract-

ed

ed in the ſame proportion. Now if the breadth of one of theſe pencils be leſs than $\frac{1}{5}$ or $\frac{1}{6}$ of a line, that is leſs than $\frac{1}{60}$ or $\frac{1}{72}$ part of an inch, the out-lines of the pictures are ſpoiled, for ſome unknown reaſon in the make of the eye, whether in the choroid, or in the retina, or in the humors it is uncertain. For by looking through an hole, in a thin plate, narrower than $\frac{1}{5}$ or $\frac{1}{6}$ of a line, the edges of objects begin to appear confuſed and ſo much the more as the hole is made narrower. Now it is eaſy to ſhew in the laſt mentioned teleſcope that the cylinder of rays is too ſlender. For by adding $\frac{1}{10}$ of the aperture to it ſelf[a], the focal diſtance of the eye- a Art. 355.
glaſs becomes $\sqrt{\frac{9}{10}} + \frac{1}{10}\sqrt{\frac{9}{10}}$, that is $\frac{11}{10}\sqrt{\frac{9}{10}}$ of an inch; and by ſimilar triangles ſubtended at the common focus q by the aperture and cylinder ſought[b], it is as the focal diſtance of the object-glaſs, to the focal diſtance b Fig. 181.
of the eye-glaſs, ſo the breadth of the aperture, to the breadth of the cylinder; that is as 30 feet or 360 inches to $\frac{11}{10}\sqrt{\frac{9}{10}}$ inches, ſo is $\sqrt{\frac{9}{10}}$ of an inch to $\frac{11}{4000}$ inch or almoſt $\frac{1}{30}$ of a line; which is much leſs than $\frac{1}{6}$. But in the teleſcope regulated in the table, it is as 360 to $3\frac{1}{10}$ ſo 3 to $\frac{11}{400}$ of an inch or almoſt $\frac{1}{3}$ of a line for the breadth of that cylinder; which can poſſibly do no harm. Hence we learn that the breadth of the aperture and focal diſtance of the eye-glaſs cannot be contracted much more than $\frac{1}{3}$ of themſelves; for even then the breadth of the cylinder at the eye will not much exceed $\frac{1}{5}$ of a line. The ſame is to be underſtood of teleſcopes of all lengths regulated as in the table, the breadth of the cylinder being the ſame in all. For by the proportion juſt mentioned it equals the breadth of the aperture multiplied into the focal diſtance of the eye-glaſs and divided by the focal diſtance of the object-glaſs, and conſequently it is proportionable to the linear aperture directly and the linear amplification inverſely; which two ratios muſt compound a ratio of equality to preſerve the ſame apparent brightneſs, by art. 349.

359. Hence though we transferred one of theſe teleſcopes from Saturn to Venus which is 225 times brighter, being 15 times nearer to the Sun, yet the breadth of the aperture muſt not be contracted above $\frac{1}{3}$ part of the whole; and if too much light ſtill remains, it muſt be diminiſhed by darkening the eye-glaſs with the ſmoak of a candle. For a greater contraction of the aperture is hurtful for another reaſon, that all the little bubbles and veins in the eye-glaſs become more conſpicuous by intercepting the whole or a greater part of thoſe little cylinders above mentioned, and conſequently the particles of the object they came from.

360. Upon the whole I conclude we may lengthen our teleſcopes at pleaſure, according to the laws of the table, with good ſucceſs; ſince not only the brightneſs and diſtinctneſs remain unaltered, but alſo the breadth of the pencils that enter the eye. Laſtly to obſerve exceeding ſmall ſtars and eſpecially the Satellites of Jupiter and Saturn, the beſt way is to increaſe very much both the aperture and focal diſtance of the eye-glaſs.

For since they appear like points even through the telescope, there is nothing gained by endeavouring to increase their diameters; but their brightness must be increased as much as possible; and this is chiefly done by increasing the aperture. By doubling its breadth, the light received into it becomes quadruple, and then by doubling also the focal distance of the eye-glass, the distinctness returns to the same as at first [a]. But still the brightness will not become 16 times greater, according to cor. 2. prop. 3, but only 4 times; because as I said the picture of the star upon the retina is but a sensible point, whose brightness cannot therefore be increased by a diminution of its breadth, but only by an addition of new light. The case is different when we view the moon and primary planets through the same telescope, whose several parts receive 16 times more light than before. Thus by widening the apertures we very much increase the power of the telescope for finding out small stars and the satellites of Saturn, so that perhaps with a 30 foot glass, whose aperture is 6 inches or double the usual one, as much may be done as with another of 120 foot whose aperture by the table is also 6 inches. So far from *Hugens*.

[a] Art. 347.

Proposition VI.

361. *In reflecting telescopes of various lengths a given object will appear equally bright and equally distinct, when their linear apertures and also their linear amplifications are as the square-square roots of the cubes of their lengths: and consequently when the focal distances of their eye-glasses are also as the square-square roots of their lengths.*

Put A for the linear aperture of the reflecting concave, L for its focal distance or the length of the telescope, F for the focal distance of the eye-glass; and when the distinctness is given A^3 is as FLL*; and when the brightness is given the amplification or $\frac{L}{F}$ is as A*, that is F is as $\frac{L}{A}$. Therefore when the distinctness and brightness are both given, A^3 is as $\frac{L^3}{A}$; or A^4 as L^3; or A as $\sqrt[4]{L^3}$. The amplification $\frac{L}{F}$ was as A, that is as $\sqrt[4]{L^3}$; and therefore F is as $\frac{\sqrt[4]{L^4}}{\sqrt[4]{L^3}}$ or $\sqrt[4]{L}$. Q. E. D.

* Art. 353.
* Art. 349.

362. In the reflecting telescope made and described by *John Hadley* Esq; F. R. S. in the Philosophical Transactions N°. 376 and 378, $L = 62\frac{1}{2}$ inches, $F = \frac{1}{3}$ or $\frac{3}{10}$ or $\frac{11}{40}$ of an inch. For he uses 3 eye-glasses and as many apertures for the reflecter whose breadths are $4\frac{1}{2}$, 5, $5\frac{1}{2}$ inches. Hence the linear amplifications or $\frac{L}{F}$ are $187\frac{1}{2}$, $208\frac{1}{3}$, $227\frac{3}{11}$ respectively. Taking

king the middle eye-glaſs and aperture for a ſtandard I computed the following table for teleſcopes of other lengths by this Rule. Call the number of inches in the length of any teleſcope L, and the focal diſtance of its eye-glaſs will be equal to $60\sqrt[4]{10L}$ in thouſandth parts of an inch. The quotient of L divided by $60\sqrt[4]{10L}$ or F gives the amplification[a], which multiplied by 24 will always give the linear aperture in thouſandth parts of an inch. For by the propoſition $\sqrt[4]{L}$ is as F; that is $\sqrt[4]{62\frac{1}{2}}$ or $\sqrt[4]{\frac{125}{2}}$ or $\sqrt[4]{\frac{625}{10}}$ or $5\sqrt[4]{\frac{1}{10}}$ is to $\sqrt[4]{L}$ as $\frac{3}{10}$ or 300 milleſimals in the given eye-glaſs, to the milleſimals in the correſpondent eye-glaſs or in $F = 60\sqrt[4]{10L}$. And the aperture being as the amplification by the propoſition; ſay, as the amplification given or $208\frac{1}{3}$ is to $\frac{L}{F}$, the amplification found, ſo is 5 inches, the aperture given, to the aperture ſought $= \frac{5}{208\frac{1}{3}} \times \frac{L}{F} = \frac{24}{1000} \times \frac{L}{F}$ inches.

a Art. 125.

363. Were it not for the unequal refrangibility of rays, refracting teleſcopes, though not ſo ſhort as theſe[b], would alſo be proportioned by this rule[c]: which not agreeing with experience, ſhews again that the aberration ariſing from the ſpherical figure are inconſiderable in compariſon to the other aberrations ariſing from the unequal refrangibility of the rays.

b Art. 341.
c Art. 333, 338.

Length of the telescope or focal dist. of the object-glass. Feet.	Linear aperture of the object-glass. Inch. & Dec	Focal dist. of the eye-glass. Inch. & Dec.	Linear amplification, or magnifying power.
1	0. 55	0. 61	20
2	0. 77	0. 85	28
3	0. 95	1. 05	34
4	1. 09	1. 20	40
5	1. 23	1. 35	44
6	1. 34	1. 47	49
7	1. 45	1. 60	53
8	1. 55	1. 71	56
9	1. 64	1. 80	60
10	1. 73	1. 90	63
13	1. 97	2. 17	72
15	2. 12	2. 32	77
20	2. 45	2. 70	89
25	2. 74	3. 01	100
30	3. 00	3. 30	109
35	3. 24	3. 56	118
40	3. 46	3. 81	126
45	3. 67	4. 04	133
50	3. 87	4. 26	141
55	4. 06	4. 47	148
60	4. 24	4. 66	154
70	4. 58	5. 04	166
80	4. 90	5. 39	178
90	5. 20	5. 72	189
100	5. 48	6. 03	199
120	6. 00	6. 60	218
140	6. 48	7. 13	235
160	6. 93	7. 62	252
180	7. 35	8. 09	267
200	7. 75	8. 53	281
220	8. 12	8. 93	295
240	8. 48	8. 83	308
260	8. 83	9. 71	321
280	9. 16	10. 08	333
300	9. 49	10. 44	345
400	10. 95	12. 05	398
500	12. 25	13. 47	445
600	13. 42	14. 76	488

Length of the telescope or focal dist. of the concave. Feet.	Focal dist. of the eye-glass. Millef. Inch.	Linear amplification, or magnifying power.	Linear aperture of the concave-metal. Millef. Inch.
$\frac{1}{2}$	0. 167	36	0. 864
1	0. 199	60	1. 440
2	0. 236	102	2. 448
3	0. 261	138	3. 312
4	0. 281	171	4. 104
5	0. 297	202	4. 848
6	0. 311	232	5. 568
7	0. 323	260	6. 240
8	0. 334	287	6. 888
9	0. 344	314	7. 536
10	0. 353	340	8. 160
11	0. 362	365	8. 760
12	0. 367	390	9. 360
13	0. 377	414	9. 936
14	0. 384	437	10. 488
15	0. 391	460	11. 040
16	0. 397	483	11. 592
17	0. 403	506	12. 143

365. These proportions, in *Hugens*'s table for refracting telescopes, are measured by the Rheinland foot which is to the English foot as 139 to 135; so that taking their lengths of as many english feet, their apertures and eye-glasses and linear amplifications should be severally diminished in the subduplicate ratio of 139 to 135 by art. 354. that is nearly in the ratio of 139 to 137 or about $\frac{1}{69}$ or $\frac{1}{70}$ part of the whole.

CHAPTER VIII.

Contains general properties of focuses and images, belonging to the eye and to any number of mediums: with general constructions, shewing the variations of the apparent distance of an object, and of the real distance of its last image, from the eye, caused by a direct motion of the eye, object or mediums.

PROPOSITION I.

HAVING *the diameters and positions of two spherical surfaces, which intercede three given mediums, and supposing the incident rays in either of the outward mediums to be parallel, and very near to the common axis of the surfaces, it is proposed to find their focus after both refractions.* Fig. 383.

366. In the common axis AC of the surfaces AB, CD, let a and d be the focuses of rays, which before they emerged by refractions at AB and CD into the outward mediums, went both ways parallel to the axis in the inward medium. Moreover let b and c be the focuses of other rays, which before they emerged by refractions at AB and CD into the inner medium, came both ways parallel to the axis in the outward mediums. These focuses may be found by art. 224. Then say as $cb : bA :: Aa : aI$; and placing aI the contrary way from a to that of bc from b, the point I shall be the focus of the rays after both refractions, which came parallel from without upon the surface CD[a]; because c is their focus after their first refraction. In like manner say as $bc : cC :: Cd : dK$, and placing dK the contrary way from d to that of cb from c, the point K shall be the focus of other rays after both refractions, which came parallel from without upon the surface AB. a Art. 237.

All the figures are adapted to mediums whose densities are continually greater as they lye in order from the left hand to the right; but the demonstrations serve for any irregular order of densities.

PROPOSITION II.

The focus of incident rays being given, it is proposed to find their focus after their refractions at two spherical surfaces, which intercede given mediums.

367. In the common axis AC of the given surfaces AB, CD, let a and d be the focuses of rays, which before they emerged by refractions into the outward mediums, went both ways parallel to the axis in the inner medium. Moreover let I and K be the focuses of other rays, which before Fig. 384.

fore their refractions through both the surfaces, came both ways parallel to the axis in the outward mediums. Then let P be the given focus of the incident rays, I and a the focuses of those rays which came the contrary way to the incident rays, and say as PI to Ia so dK to KR; and placing KR the contrary way from K to that of IP from I, the point R shall be their focus after both the refractions.

Fig. 385. For by saying as $Pa : aA :: Ab : bQ$ and by placing bQ the contrary way from b to that of aP from a, the point Q is their focus after the first

a Art. 237. refraction at AB[a]. The same point Q being the focus of incident rays upon the surface CD, say again as $Qc : cC :: Cd : dR$; and placing dR the contrary way from d to that of cQ from c, the point R is their focus after both refractions. Now by the first of these proportions, and of those in the foregoing proposition, the rectangle $Pa \times bQ = (aA \times Ab =)\ bc \times Ia$; and by the second proportions in this and in the foregoing proposition, the rectangle $Qc \times dR = (cC \times Cd =)\ bc \times dK$; then by resolving the two former rectangles into the proportion of their sides, it is as $Ia : Pa :: bQ : bc$ and disjointly (or conjointly) $PI : Pa :: (Qc : bc ::)\ dK : dR$ by resolving the two latter rectangles; and disjointly (or conjointly) $PI : Ia :: dK : KR$.

368. *Corol.* The magnitude of the rectangle under PI, KR is invariable, being always equal to the given rectangle under Ia, dK; and consequently KR is reciprocally as PI.

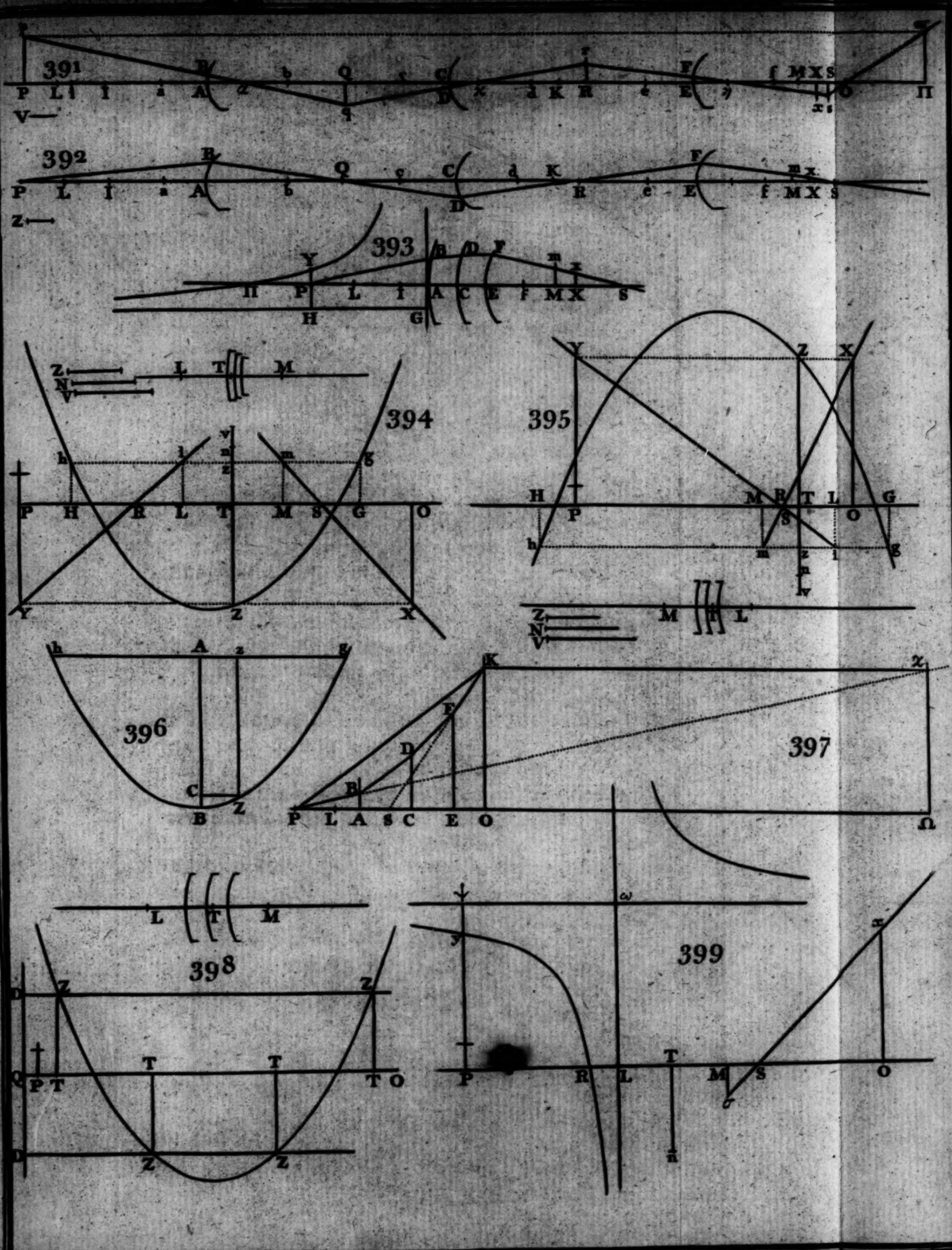
PROPOSITION III.

Having the diameters and positions of three spherical surfaces, which intercede four given mediums, if the incident rays in either of the outward mediums be parallel, and very near to the common axis of the surfaces, it is proposed to find their focus after all the refractions.

Fig. 386. 369. Supposing the focuses a, d and I, K to be determined by the first proposition for two contiguous surfaces, as AB and CD; let parallel rays falling on both sides of the third surface EF have their focuses, after refraction there only, at e and f; whereof e being the focus of incident rays upon the surface CD, say as $eK : Kd :: aI : IL$; and placing IL the contrary way from I to that of Ke from K, by the foregoing proposition the point L is the focus of rays, after all the refractions, which came parallel from without upon the surface EF. Again, say as $Ke : eE :: Ef : fM$, and placing fM the contrary way from f to that of eK from e, the point M is the focus of rays after all the refractions which came pa-

b Art. 237. rallel from without upon the surface AB[b].

PROPO-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PROPOSITION IV.

The focus of incident rays being given, it is proposed to find their focus after refractions through any given number of spherical surfaces that intercede given mediums.

370. *Case* 1. In the common axis of three surfaces AB, CD, EF, let I and f be the focuses of rays, which before they emerged by refractions into the outward mediums, went both ways parallel to the axis in either of the inner mediums, as CE; moreover let L and M be the focuses of other rays, which before their refractions through all the surfaces, came both ways parallel to the axis in the outward mediums; then let P be the given focus of the incident rays, L and I the focuses of those rays which came the contrary way to the incident rays, and say as $PL:LI::fM:MS$; and placing MS the contrary way from M to that of LP from L, the point S shall be their focus after three refractions. Fig. 387.

For by the second proposition their focus R, after two refractions at the surfaces AB, CD, is given by saying as $PI:Ia::dK:KR$; and since R is their focus when incident on the surface EF, say as $Re:eE::Ef:fS$, and S is their focus after three refractions [a]. Now by the first of these proportions and of those in the third proposition, the rectangle $PI\times KR=(Ia\times dK=)\,Ke\times LI$. Likewise by the second of these proportions and of those in the third proposition, the rectangle $Re\times fS=(eE\times Ef=)\,Ke\times fM$; and by resolving the two former rectangles into the proportion of their sides, it is as $LI:PI::KR:Ke$, and disjointly (or conjointly) $PL:PI::(Re:Ke::)\,fM:fS$ by resolving the latter rectangles; and disjointly (or conjointly) $PL:LI::fM:MS$. Fig. 388. a Art. 237.

The rays whose focuses are I and f were supposed to be parallel to the axis in the medium CE; now let a and t be the focuses of other rays which are parallel to the axis in the other inner medium AC; and since $PL:LI::fM:MS$; for the same reason when P comes to a and consequently S to t, it is as $aL:LI::fM:Mt$, therefore because of the given rectangle $LI\times fM$, it is as $PL:La::tM:MS$. Fig. 389, 390.

371. *Case* 2. Hence by the method of the third proposition one may find the focus of parallel rays after refraction through four surfaces, and then the focus of inclined rays in the same manner as in the foregoing case, and so on. And by these propositions it is sufficiently evident, that if L and M be the principal focuses of the whole system of surfaces, and I and f be the focuses of other rays that go parallel in any one of the inner mediums, then $PL:LI::fM:MS$.

372. *Corol.* 1. It is manifest by the analogy between the rules for finding the focuses of rays refracted through a single surface and a single lens [b], that the rule in this proposition for the focuses of rays refracted through any number of surfaces, will serve also for any number of lenses of any sort placed b Art. 236.

placed at A, C, D, &c. in one continued medium. And so it appears that the relation between the conjugate focuses P, S, of a pencil of rays refracted through an infinite number of surfaces or lenses, may be always expressed by a single proportion; in like manner as that relation is expressed in a single surface or a single lens.

373. *Corol.* 2. Take a line N a middle proportional between the given lines LI, fM, or between La, tM, and it will also be a middle proportional between the variable lines PL, MS: and consequently PL is reciprocally as MS, and they lye contrary ways from the principal focuses L, M. And the emergent rays will flow from S if it lyes on the same side of the last surface as the incident rays, otherwise towards S. Because the rays continue to go forwards from the last surface.

PROPOSITION V.

Fig. 391. *Having the semidiameter* Pp *of a small object, placed perpendicularly to the common axis of any number of refracting surfaces, which intercede given mediums; it is proposed to find the semidiameter* Ss *of its last image.*

374. Things remaining as they were in the preceding propositions, take a line $V = Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$ &c. as far as the number of surfaces permits; and Ss will be to Pp as the constant line V to PL.

For let α, $\varkappa$, η, &c. be the centers of the given surfaces A, C, E, &c; Qq, Rr, Ss, &c. the respective images formed by them. The object and these images are terminated by the lines $p\alpha q$, $q\varkappa r$, $r\eta s$, &c[a]. Now by art. 238. it is as $Qb : QA :: Q\alpha : QP$, and disjointly (or conjointly) $Qb : bA :: Q\alpha : \alpha P :: Qq : Pp$, because the triangles $Q\alpha q$, $P\alpha p$ are equiangular; and for the same reason that $Pp : Qq :: bA : bQ$, it is as $Qq : Rr :: dC : dR$, and as $Rr : Ss :: fE : fS$, &c. And by compounding these proportions, it is as $Pp : Ss :: bA \times dC \times fE : bQ \times dR \times fS$. But by art. 237. it will be as $Pa : Ab :: Aa : bQ$, and by art. 367. as $PI : Pa :: dK : dR$. and by art. 370. as $PL : PI :: fM : fS$; and these three proportions being compounded, give $PL : Ab :: Aa \times dK \times fM : bQ \times dR \times fS$. But we had $Ss : Pp :: bQ \times dR \times fS : bA \times dC \times fE$; therefore $Ss \times PL : Pp \times Ab :: Aa \times dK \times fM : bA \times dC \times fE$. Whence $Ss = Pp \times \frac{Aa}{PL} \times \frac{dK}{dC} \times \frac{fM}{fE}$; or, by taking a line $V = Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$, $Ss = Pp \times \frac{V}{PL}$.

[a] Art. 245.

375. *Corol.* 1. Hence the image Ss is equal to the object Pp, when PL is equal to V.

376. *Corol.* 2. The image Ss is as the object Pp directly and as PL inversely.

377. *Corol.* 3. And therefore the image Ss is as the angle PLp which the object subtends at the principal focus L[b].

[b] Art. 222.

378.

378. *Corol.* 4. The image is similar to the object in all its parts. For when PL is given, Ss is as Pp.

379. *Corol.* 5. When the object is given, the image is inversely as PL or directly as MS by art. 373.

380. *Corol.* 6. If the emergent rays be received upon a perpendicular plane cutting ηS in X and ηs in x, take a line $Ll = \frac{LI \times fM}{\eta M}$ and place it in a contrary direction to that of $M\eta$ from M, and the semidiameter Xx of this confused image will be equal to $\frac{Pp}{PL} \times \frac{\eta X}{\eta M} \times V$. For the triangles ηXx, ηSs being similar, it is as $Xx : (Ss) \frac{Pp}{PL} \times V^* :: \eta X : \eta M + MS$, * Art. 374.
or $\eta M + \frac{LI \times fM^*}{PL} :: \eta X \times PL : \eta M \times PL + LI \times fM :: \frac{\eta X \times PL}{\eta M} : PL$ * Art. 373.
$+ \frac{LI \times fM}{\eta M}$ or $PL + Ll$ or Pl by construction. Whence $Xx = \frac{Pp}{Pl} \times \frac{\eta X}{\eta M} \times V$.

381. *Corol.* 7. Hence if the perpendicular plane be fixt at any place X, the confused image Xx is as $\frac{Pp}{Pl}$, or as the angle subtended by the object at the given point l^*. * Art. 222.

Corol. 8. Hence if the object be given, this image Xx is inversely as Pl; and is also similar to the object.

PROPOSITION VI.

It is proposed to find the ratio of the angles which the incident and emergent parts of a ray do make with each other and with the common axis of any number of given surfaces betwixt given mediums.

382. Things remaining as they were, let $PBDFS$ be the course of the ray; P and S its first and last intersections with the common axis of the surfaces. Take a line Z equal to $Ab \times \frac{dK}{cC} \times \frac{fM}{eE}$ &c. as far as the number of surfaces permits; and the angle APB will be to ESF as the given line Z to PL. Fig. 392.

For $\mathcal{Q}b : bA :: Aa : aP^*$ and conjointly and alternately $\mathcal{Q}b : Aa ::$ * Art. 237.
$\mathcal{Q}A : AP ::$ ang. APB : ang. $A\mathcal{Q}B^*$. In like manner, $Rd : Cc :: RC :$ * Art. 222.
$C\mathcal{Q} ::$ ang. $C\mathcal{Q}D$: ang CRD; and likewise $Sf : Ee :: SE : ER ::$ ang. ERF : ang. ESF; and so on. And by compounding these proportions it is as $\mathcal{Q}b \times Rd \times Sf : Aa \times Cc \times Ee ::$ ang. P : ang. S. But in the demonstration of the last proposition we had $Aa \times dK \times fM : \mathcal{Q}b \times Rd \times Sf :: PL : Ab$. Whence it easily follows that $Ab \times \frac{dK}{Cc} \times \frac{fM}{Ee} : PL ::$ ang. P : angle S. And by substracting the lesser angle from the greater we have

the angle made by the incident and emergent ray, as appears by producing these rays till they cross one another.

383. *Corol.* 1. Let AB be the semiaperture of the first surface, and the semidiameter Mm of a perpendicular section of the emergent pencil of rays will be to AB, as the given line $\frac{LI \times fM}{Z}$ is to AP. For because the subtenses of small angles are in a compound ratio of their legs and of the angles themselves, we have Mm to AB in a ratio compounded of MS to AP, and of the angle MSm to the angle APB, or, by the proposition, of PL to Z.

* Art. 371. Whence $Mm = \frac{AB}{AP} \times \frac{MS \times PL}{Z} = \frac{AB}{AP} \times \frac{LI \times fM}{Z}$*.

384. *Corol.* 2. And therefore Mm is as $\frac{AB}{AP}$, or as the angle APB.

385. *Corol.* 3. So that when AB is given, Mm is reciprocally as AP.

386. *Corol.* 4. If the emergent pencil be cut by a perpendicular plane at any other place X, the semidiameter Xx, of this section, is equal to $\frac{AB}{AP} \times \frac{LI \times fM - PL \times MX}{Z}$. For $Xx : (Mm) \frac{AB}{AP} \times \frac{LI \times fM}{Z} :: MS - MX : MS :: \frac{LI \times fM}{PL} - MX : \frac{LI \times fM}{PL} :: LI \times fM - PL \times MX : LI \times fM$.

Fig. 393. 387. *Corol.* 5. Hence we have the following construction, take $L\Pi : LI :: fM : MX$ and place it in a contrary direction from L to that of MX from M; and in the line AG perpendicular to the axis, take AG to AB as MX to Z; then through G draw GH parallel to the axis, and with the asymptotes GA, GH draw an hyperbola ΠY through the point Π; and the perpendicular PY will every where equal the semidiameter Xx when the plane is fixt at X. For by the property of the hyperbola the rectangle $GH \times HY = GA \times A\Pi$, whence $PY = (HY - GA =) \frac{GA \times A\Pi - GA \times AP}{AP} = \frac{AG}{AP} \times \overline{L\Pi - LP} = \frac{AG}{AP} \times \overline{\frac{LI \times fM}{MX} - LP} = \frac{AG}{AP} \times \frac{LI \times fM - LP \times MX}{MX} = \frac{AB}{AP} \times \frac{LI \times fM - LP \times MX}{Z}$ (by construction) $= Xx$ in the 4th corol.

388. The line N or a middle proportional between LI and fM was made use of to determine the relation of any two conjugate focuses P, S,

a Art. 373. to the principal focuses L, M[a]; the line V or $Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$, to deter-

b Art. 374. mine the ratio of the object at P to its image at S[b]; and the line Z or $Ab \times \frac{dK}{cC} \times \frac{fM}{eE}$, to determine the ratio of the angles at P and S made by any

c Art. 382. ray with the axis of the surface[c]; I say the lines V, N, Z are continual proportionals in a subduplicate ratio of $Aa \times Cc \times Ee$ to $Ab \times Cd \times Ef$; and consequently in lenses they are equal to one another. For $V : Z :: \frac{Aa}{Cd \times Ef} : \frac{Ab}{Cc \times Ee} :: Aa \times Cc \times Ee : Ab \times Cd \times Ef$; and by art. 369 and

366 it will appear that $N^2 : Z^2$ or $LI \times fM : \frac{Ab^2}{Cc^2 \times Ee^2} \times dK^2 \times fM^2 ::$ $(LI)\ \frac{Ia \times dK}{Ke} : \frac{Ab^2}{Cc^2 \times Ee^2} \times dK^2 \times \frac{eEf}{Ke} :: (Ia)\ \frac{aAb}{bc} : \frac{Ab^2}{Cc^2 \times Ee} \times \frac{cCd}{bc} \times Ef$ $:: \frac{Aa}{Cd \times Ef} : \frac{Ab}{Cc \times Ee} :: V : Z$, as before.

Proposition VII.

To find the apparent diſtance of an object ſeen through any given ſyſtem of mediums; and to ſhew how it varies while the eye, object or ſyſtem is moved forward or backward.

389. Things remaining as they were, biſect LM in T, and in a perpendicular to the axis at T, take Tv, Tn, Tz ſeverally equal to the given lines V, N, Z; and through z, parallel to LM, draw lm cutting perpendiculars at L and M in l and m. Join mS and produce it both ways, and a perpendicular OX terminated at mS will be equal to the apparent diſtance of the object P ſeen from any point O. Fig. 394, 395.

In like manner let R be the conjugate focus of a pencil of rays ſuppoſed to flow from O; join lR and produce it, and the perpendicular PY terminated at lR will alſo be the apparent diſtance of the object P ſeen from O.

Laſtly take OG and PH each equal to TL or TM and place them inwards, if the order of the points LTM be according to the courſe of the rays that flow from P, otherwiſe outwards; and let lm produced cut the perpendiculars to the axis at G and H in g and h; then let gh be an ordinate to the axis of a parabola gZh, whoſe parameter belonging to the axis is the line Tv, and whoſe legs are extended from its vertex the ſame way as the perpendiculars Gg, Hh are extended from the axis of the ſyſtem; and the ordinate TZ will alſo be the apparent diſtance of the object P ſeen from O.

Now if the object and the ſyſtem be fixt while the eye is in motion along the axis of the ſyſtem, the moveable perpendicular OX, terminated at the fixt line mS, being always equal to the apparent diſtance, will ſhew how it varies. Or if the eye and the ſyſtem be fixt while the object is in motion along the axis of the ſyſtem, the moveable perpendicular PY, terminated at the fixt line lR, being always equal to the apparent diſtance, will ſhew how it varies. Or laſtly if the eye and object be fixt while the ſyſtem is in motion along its own axis, (ſuppoſing its parts to be kept at the ſame intervals,) the moveable perpendicular TZ terminated at the fixt parabola rZs, being always equal to the apparent diſtance, will ſhew how it varies.

And if one part of the ſyſtem be fixt, while the other is in motion, let the fixt image of the object formed by the fixt part, be at P; and by apply-

ing the same construction to the moveable part as before was applyed to the whole, the ordinate TZ will shew the apparent distance in this case.

DEMONSTRATION.

Fig. 391. 390. For let the line $p\varpi$ drawn parallel to PO meet the visual ray Os produced, in ϖ, and compleat the rectangular parallelogram $Pp\varpi\Pi$; then $O\Pi$ is the apparent distance of the object Pp [a]; and the triangles $O\Pi\varpi$, OSs being equiangular, we have $O\Pi : OS :: (\Pi\varpi$ or $Pp : Ss ::) PL : V$ [b], that is supposing the perpendiculars OX, PY, TZ to be severally equal to the apparent distance $O\Pi$, we have OX or PY or $TZ : OS :: PL : Tv :: Tz : MS$; because $PL \times MS = Tn^2 = Tv \times Tz$ [c].

a Art. 139.
b Art. 374.
Fig. 394, 395.
c Art. 373. 388.

First then we have $OX : OS :: Tz$ or $Mm : MS$, which shews that mS produced is the geometrick Place of the point X. Secondly we have also $PY : Tz$ or $Ll :: (OS : MS ::) PR : LR$; for we had $PL \times MS = Tn^2 = OM \times LR$, whence $OM : MS :: PL : LR$ and disjointly $OS : MS :: PR : LR$. And this shews that lR produced is the place of the point Y. Lastly we have $TZ = \left(\frac{PL \times OS}{Tv} = \frac{PL \times OM}{Tv} - \frac{PL \times MS}{Tv} = \right) \frac{HT \times TG}{Tv} - Tz$. For by construction $PL = HT$ and $OM = TG$ and $PL \times MS = Tv \times Tz$. Now let the point T be removed to G; then because $TG = o$, by the equation we have the ordinate $TZ = -Tz = Gg$. Again by removing T to H, we have $TZ = -Tz = Hh$; and by the same equation we have

Fig. 396. $TZ + Tz$ that is $zZ = \frac{hz \times zg}{Tv}$. Bisect gh in A, and the point z being removed to A, let zZ become $AB = \frac{hAg}{Tv} = \frac{hA^2}{Tv}$. Draw ZC perpendicular to AB, then $BC = \left(AB - zZ = \frac{hA^2}{Tv} - \frac{hA^2 - Az^2}{Tv}\,^{*} = \frac{Az^2}{Tv} = \right) \frac{CZ^2}{Tv}$. Which shews that the place of the point Z is a parabola whose parameter belonging to the axis AB is Tv. Q. E. D.

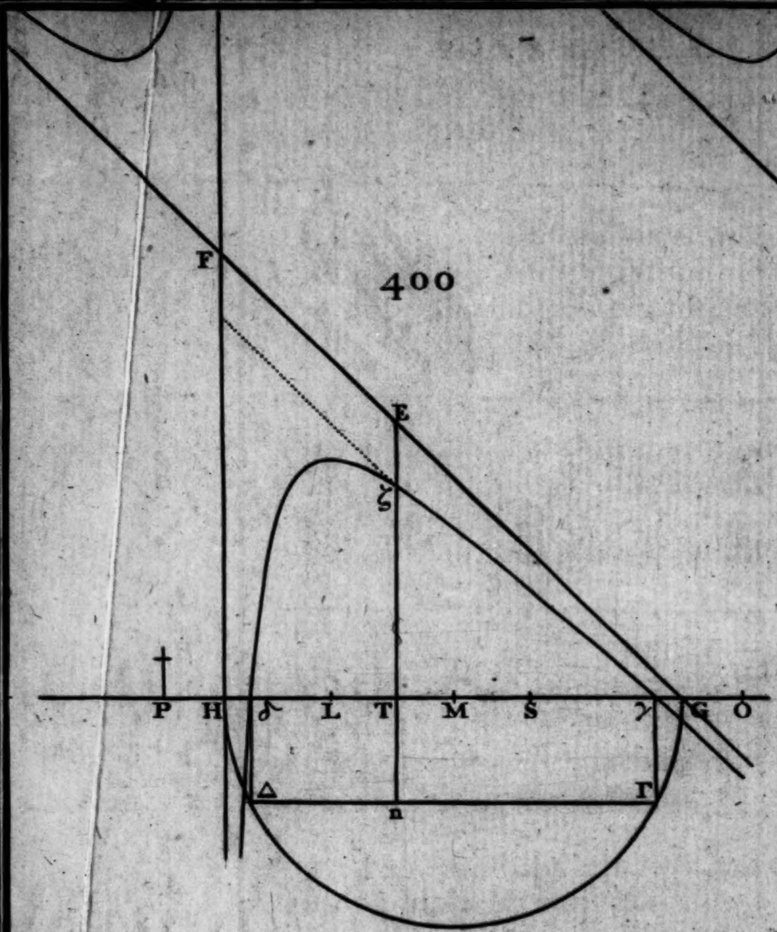
* Euc. II. 5.

Fig. 397. 391. While the system remains fixt in any place, the apparent distance of an object at P seen from O, will be to the apparent distance of an object at O seen from P, as Z to V. For let a ray $PBDF$ fall upon an object at O in any point K, draw $K\varkappa$ parallel to OP and let it meet the ray PB, produced, in $\varkappa$, and compleat the rectangle $\varkappa KO\Omega$; then the apparent distance $P\Omega : PO ::$ angle OPK : angle $OP\varkappa$*, or in a ratio compounded of the angle OPK to OSK and of OSK to $\Omega P\varkappa$, that is of OS to OP and of PL to Z*. Therefore $P\Omega = \frac{PL \times OS}{Z}$. But we had $O\Pi = \frac{PL \times OS}{V}$ in the article above, and consequently $O\Pi : P\Omega :: Z : V$.

* Art. 222.
* Art. 382.

Fig. 394, 395. 392. Hence the foregoing construction will give the apparent distance of an object at O seen from P, by drawing the line lm through v and by making

400



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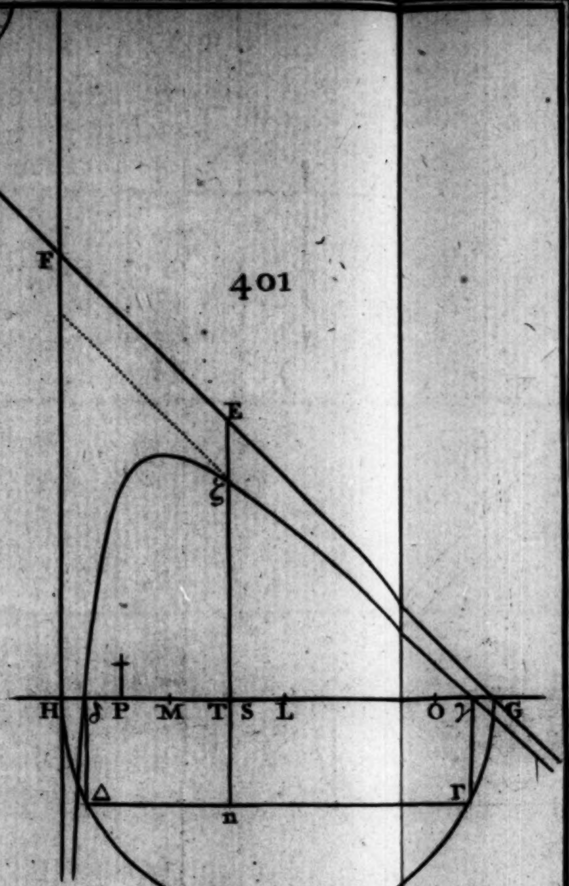

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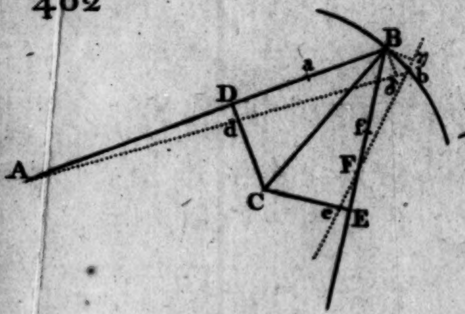

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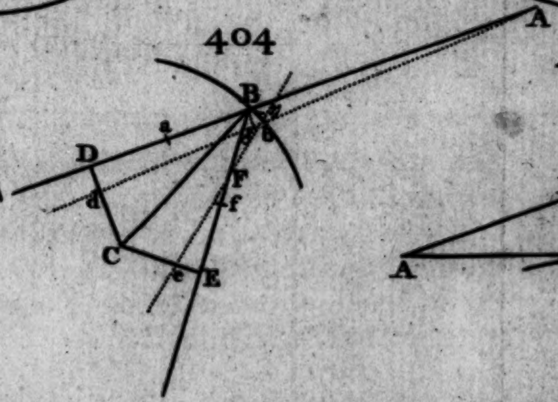

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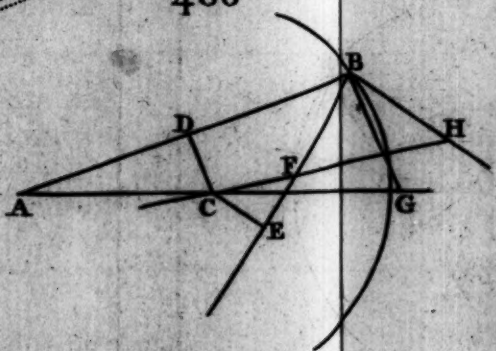

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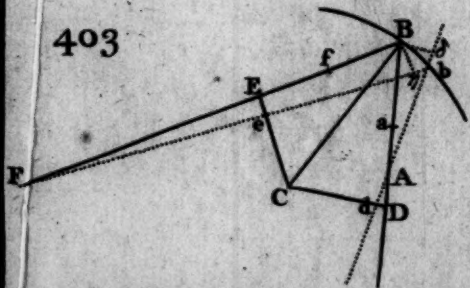

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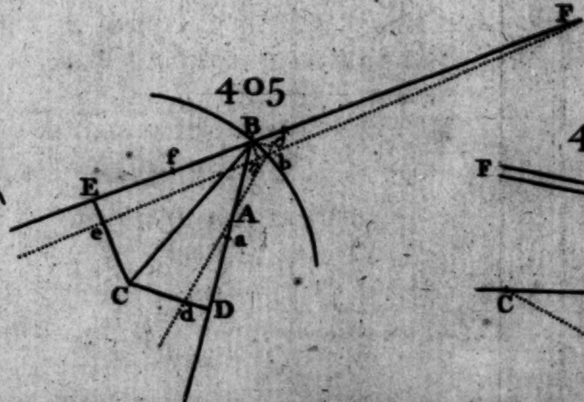

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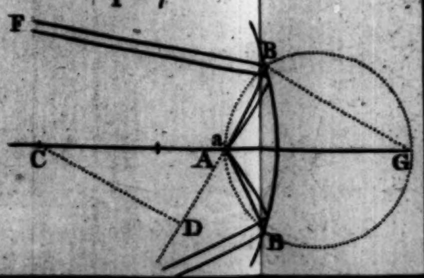


making Tz the parameter of the parabola. The ſame conſtruction ſerves alſo for a lens or lenſes, by drawing the line lm through n, and by making Tn the parameter of the parabola. For in lenſes the points n, v, z coincide [a]; and conſequently the apparent diſtances $O\Omega$ and $P\Omega$ are equal. a Art. 388.

393. Hence if the places of the eye and object be given, and it be required to interpoſe a given ſyſtem of ſurfaces or lenſes in ſuch a place that the object ſhall appear through them all at a given diſtance; in any perpendicular DQD to the axis OPQ, on each ſide of Q take QD and QD equal to the given diſtance, and through the points D, D, parallel to the axis, draw the lines DZZ, DZZ cutting the parabola in four points Z, Z, Z, Z when poſſible; then draw the perpendiculars ZT &c. to the axis OP, and place that point of the given ſyſtem which biſects the interval LM at any of the four points T; and to the eye at O the object will appear at the given diſtance QD or TZ, as is evident by the foregoing conſtruction. Fig. 398.

394. And by the ſame method, if the places of the object and ſyſtem be given, one may find the place of the eye, from whence the object ſhall appear at a given diſtance; and likewiſe if the places of the eye and ſyſtem be given, one may find the place of the object, where it ſhall appear at a given diſtance.

I have given the foregoing conſtructions of apparent diſtance and magnitude [b], for any one to examine how near they will anſwer his ideas of diſtance and magnitude upon making the experiments. And I add the following conſtructions for the diſtance of the laſt image of an object from the eye, to ſhew how widely it will differ from the apparent diſtance of the object, in moſt caſes. b Art. 140.

Proposition VIII.

To ſhew in what manner the diſtance between the eye and the laſt image of an object is varied, while the eye, object or refracting ſyſtem of any ſort, is moved forward or backward.

395. Things remaining as they were, take a perpendicular $M\sigma$ equal to MS, and draw σS cutting a perpendicular at O in x; and when the object and the ſyſtem are fixt, and the eye is in motion along OP, it is evident that Ox will be always equal to OS, the diſtance between the eye and the laſt image of the object P. Fig. 399.

At L erect a perpendicular $L\omega$ equal to MO, and draw $\omega\psi$ parallel to LP, cutting a perpendicular at P in ψ; and through the focus R, with the center ω and aſymptotes ωL, $\omega\psi$ draw an hyperbola cutting the perpendicular at P in y; and Py will be the diſtance of the laſt image from the eye, while the eye and ſyſtem are fixt and the object is in motion along OP. For ſince $P\psi = L\omega = MO$, we ſhall have $Py = OS$ by taking $\psi y = \left(MS = \frac{Tn^2}{PL} =\right) \frac{Tn^2}{\omega\psi}$. Hence the rectangle $\omega\psi y = Tn^2$, which ſhews

that

that the place of the point y is an hyperbola; which will paſs through R, becauſe the rectangle $RL\omega$ or $RL \times MO = Tn^2$.

Fig. 400, 401. Laſtly erect a perpendicular $HF = HG$ and draw FG cutting the perpendicular nT produced in E, in which take $E\zeta = \frac{Tn^2}{HT}$ and place it downward if HT lyes from H toward G, otherwiſe upward, and through the point ζ, with the center F and aſymptotes FG, FH, deſcribe an hyperbola $\gamma\zeta\delta$; and while the eye and object are fixt, and the point T is in motion along with the ſyſtem, the ordinate $T\zeta$ will be always equal to the diſtance of the laſt image from the eye. For by conſtruction the line $E\zeta$ is reciprocally as HT, or reciprocally as FE, becauſe HT is to FE in a given ratio of HG to FG; that is, the magnitude of the rectangle under $FE\zeta$ is invariable, and conſequently the place of the point ζ is an hyperbola. Moreover by conſtruction $TE = TG = MO$, and therefore $T\zeta = (TE - E\zeta = MO - \frac{Tn^2}{HT} = MO - \frac{Tn^2}{PL} = MO - MS =) OS$.

396. Let the hyperbola $\gamma\zeta\delta$ cut the line OP in γ and δ; and when the point T in the given ſyſtem is brought to γ or δ, the laſt image of the object at P will fall upon the eye at O; becauſe $T\zeta = o$.

397. Hence if two points O, P be given, and it be required to interpoſe a given lens or ſurface or a ſyſtem of lenſes or ſurfaces in ſuch a place, that the rays which belong to P ſhall belong to O after all the refractions, we have the ſolution in the foregoing article; or more readily in this manner. Parallel to PO, and through the point n, draw a line $\Gamma n \Delta$, cutting a ſemicircle upon the diameter GH in Γ, Δ; and the perpendiculars $\Gamma\gamma$, $\Delta\delta$ to the axis OP, will give the points γ, δ where T muſt be placed; for when OS or $T\zeta$ that is $TG - \frac{Tn^2}{TH} = o$, we have the rectangle $HTG = Tn^2$

* Euc. VI. 13. $= \gamma\Gamma^2 = \delta\Delta^2 = H\gamma G = H\delta G$*; and conſequently T muſt coincide with γ or δ. But the *data* muſt be ſuch that Tn muſt be leſs than half HG; for if it be bigger, it will be impoſſible for the line $\Gamma n \Delta$ to cut the circle whoſe diameter is HG.

398. And when the diſtance of the image from the eye is diminiſhed to nothing, the apparent diſtance of the object becomes nothing at the ſame time. For the parabola in the foregoing propoſition will cut the line OP in the ſame points γ, δ as the hyperbola does, when poſſible, that is when Tn is leſs than half GH. For we had the parabolick ordinate TZ

* Art. 390. $= \frac{HT \times TG}{Tv} - Tz$*; which being put equal to nothing gives the rectangle $Tv \times Tz$ or $Tn^2 = HTG = H\gamma G$.

399. In all the propoſitions of this chapter I have only purſued the moſt general caſe, that is when the ſyſtem of ſurfaces or lenſes has two principal focuſes, as L and M; but the ſurfaces may be ſo ſituated that the

the rays which fall parallel upon the first, shall emerge parallel from the last. These particular cases and some few determinations, which would have perplexed the reader too much and withdrawn his attention from the general case, I have purposely omitted. Whoever has a mind to consider them by themselves, will find them easy to be solved in the following manner. In the second proposition if the focuses b, c be supposed to coincide, the principal focus I, K will be removed to an infinite distance. In this case Pa is to dR in the given ratio of the rectangle aAb to the rectangle cCd; and Pa, dR will lye the same way from a and d. For we had $Pa = \frac{aAb}{bQ}$ and $dR = \frac{cCd}{cQ}$; but $bQ = cQ$, because $bc = o$. In this and the like cases when the system has no principal focuses, the geometrick places of the points X, Y, Z; x, y, ζ, in the foregoing constructions are all straight lines. Fig. 383. to 385.

400. By the analogy between the rules for finding the focus of a pencil of rays reflected and refracted at a single spherical surface [a], it is manifest that all the propositions and constructions in this chapter, may be easily adapted to rays successively reflected from any number of spherical surfaces betwixt any given mediums. And though I have mentioned none but spherical surfaces, yet all the conclusions are the same for any other surfaces of equal curvities to the spherical ones [b].

a Art. 207. 236.

b Art. 218.

401. Likewise in the following chapter, where I consider the reflections and refractions of rays that fall not only perpendicularly, or almost perpendicularly as before, but with any degree of obliquity upon any curves, I generally mention no other but a circle, which is always supposed to have the same curvity as any other curve, when possible, in the place where a slender pencil of rays is incident upon it. To find the radius of curvity, is the same thing as to draw perpendiculars to a given curve or to its tangents at the given points of incidence; and then to find the concourse of two of the nearest perpendiculars; for this point of concourse is the center of a circle whose arch coincides with the given curve at the points of incidence. But this being a problem purely geometrical, it is sufficient in this place to have given an idea of it; especially since the consideration of any other curves but circles, is seldom necessary for the solution of any optical appearances in nature. At first I shall consider a superficial pencil of rays, that lye all in one plane of incidence; because it will appear afterwards that the rays of a solid pencil, which lye in different planes of incidence, will belong to different focuses; and therefore instead of a spherical surface I substitute a great circle of that surface; considering it as a physical circle, not a mathematical one. Lastly it is to be observed as before [c], that in strictness of geometry the focus of a superficial pencil of reflected or refracted rays, is nothing but the common intersection of two contiguous rays, whose points of incidence were in the middle of all the rest; but the same intersection considered physically may be called the focus of a slender superficial pencil of rays. Design.

c Art. 211.

CHAP-

CHAPTER IX.

Determinations of focuses of rays falling with any degrees of obliquity upon any number of reflecting and refracting surfaces of any sort, and also of the properties of Causticks.

PROPOSITION I.

Fig. 402 to 405.

402. LET *the ray* AB *belonging to the focus* A, *fall with any obliquity upon the concavity or convexity of a circle or any other curve, whose radius of curvity at* B *is* CB, *and be reflected along* BF *given in position; draw the sines of incidence and reflection* CD, CE, *and bisect their equal cosines* BD, BE *in* a *and* f; *and say as* Aa *to* aB *so* Bf *to* fF, *and place* fF *the same way with respect to* Bf *as* Aa *lyes with respect to* aB; *and the point* F *will be the focus of a slender pencil reflected from an exceeding small arch whose middle point is* B.

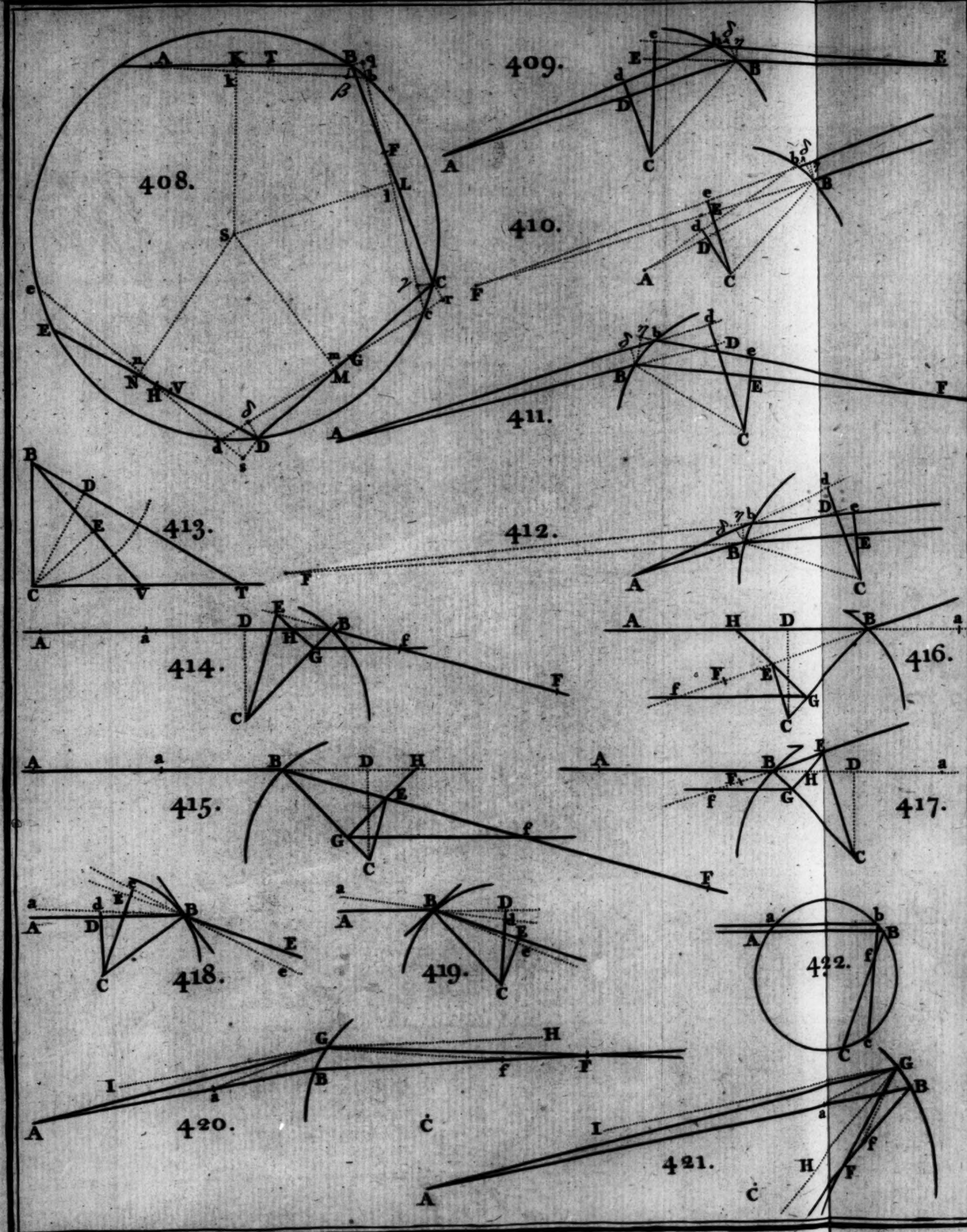
For let AbF be another ray reflected from b, the nearest point to B; draw Cd and $B\delta$ perpendicular to Ab, and also Ce and $B\eta$ perpendicular to bF; and the line $B\delta$ will equal $B\eta$. For the angles $Bb\delta$, $Bb\eta$, which Ab and bF produced do make with the arch Bb or its tangent at b, are equal[a]; and consequently the little right angled triangles $Bb\delta$, $Bb\eta$ are also equal. Again the lines Dd and Ee, which are the same at last, when b and B are coinciding, as the differences of the equal sines CD and CE, Cd and Ce, are also equal. And since the triangles $BA\delta$, DAd are equiangular, and also $BF\eta$, EFe; the ratio of the distances BA to AD (or of $B\delta$ to Dd or of $B\eta$ to Ee) is the same as the ratio of the similar distances BF to FE. Hence conjointly and disjointly $BA \pm AD : AD :: BF \pm FE : FE$, and alternately $\frac{BA+AD}{2} : \frac{BF+FE}{2} :: (AD : FE ::) \frac{BA-AD}{2} : \frac{BF-FE}{2}$, that is $Aa : Bf :: aB : fF$; and alternately $Aa : aB :: Bf : fF$. And supposing the incident rays to go backwards in these figures, they will be reflected from the convexity of the arch, in the same lines produced as before. *Q. E. D.*

a Art. 9.

403. *Corol.* 1. When the incident ray AB passes through the center of the reflecting circle, this proposition degenerates to the third proposition, art. 207.

404. *Corol.* 2. It appeared in the demonstration that the ratio of the distances BA to AD is the same as the ratio of the similar distances BF to FE.

405. *Corol.* 3. The points a, f are the focuses of slender pencils coming parallel to FB and AB respectively. For when the focus A or F is removed to an infinite distance the lines Dd, $B\delta$, $B\eta$, Ee become equal to one another


408.
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another. Or it appears from the second corollary, or from the proposition it self.

406. *Corol.* 4. Erect BG and BH perpendiculars to the incident and reflected rays AB and BF, and taking BH equal to BG, terminated at the axis AC produced, joyn CH and it will cut the reflected ray in the focus F. For drawing the equal sines CD, CE, the triangles BAG, DAC and also BFH, EFC are equiangular. Whence $BA:AD::(BG:DC::BH:CE::)\ BF:EF$; which is the property of the focuses A, F, by cor. 2. Fig. 406.

407. *Corol.* 5. The focus A being given to find the point B, in a reflecting circle, from which and from the adjoining points, the rays shall be reflected into parallel lines when possible; in CA produced take AG equal to AC, and a circle described upon the diameter AG will cut the reflecting circle in the points B, B required. For drawing GB, the angle ABG in the semicircle is a right one, and consequently in the equiangular triangles ABG, ADC, the sides AB, AD are equal; and therefore A is the focus of rays that will be reflected from B into parallel lines by cor. 3. Fig. 407.

408. The position of the reflected ray BF, which in this and the following proposition is supposed to be given, may be determined by making the angle of incidence equal to the angle of reflection, or by inscribing a chord in the reflecting circle, equal to the chord which the incident ray moved in; or by several other methods.

PROPOSITION II.

409. *The focus of incident rays being given, it is required to find their focus after any given number of successive reflections from the inside of a given circle.*

Let $ABCDE$ be the given course of a ray reflected from a circle at B, C, D; and from the center S upon AB and DE, the first and last parts of the ray, draw the perpendiculars SK, SN; and from K and N towards B and D, the first and last points of reflection, take KT and NV severally in proportion to KB or ND, as an unite to twice the number of successive reflections; then let any point A in the line AB be the focus of incident rays, and say as TA to TK so VN to VH; and placing VH the same way with respect to VN as TA lyes with respect to TK, the point H will be their focus after all the reflections. Fig. 408.

For let the perpendiculars SK, SL, SM, SN upon the ray AB, BC, CD, DE cut the next ray $Abcde$ in k, l, m, n; and let the focuses or intersections of these two rays be at F, G, H. Also from B, C, D upon the ray Ab, bc, cd, draw the perpendiculars $B\beta$, $C\gamma$, $D\delta$, and likewise upon bc, cd, de draw the perpendiculars Bq, Cr, Ds. These latter perpendiculars Bq, Cr, Ds are respectively equal to the former $B\beta$, $C\gamma$, $D\delta$; as it appeared in the demonstration of the last proposition. Now since FB,

FL, FC are in the same ratios to one another as Bq, Ll, $C\gamma$; and since $FB + 2FL = FC$, therefore $Bq + 2Ll = C\gamma$ or Cr. For the like reason $Cr + 2Mm = D\delta$ or Ds, and so on. Therefore since Kk, Ll, Mm are equal, as was shewn in the demonstration of the last proposition, the perpendiculars Bq, Cr, Ds, and their equals $B\varrho$, $C\gamma$, $D\delta$, are severally in arithmetical progression; their common difference being $2\,Ll$ or $2\,Kk$. So that putting n for the number of chords BC, CD, that join the successive points of reflection, made in the passage of the given ray from A to H, we have $B\beta + 2nKk = Ds$ the subtense of the angle DHS; that is, $Kk : B\beta + 2nKk :: (Kk$ or $Nn : Ds ::)\ NH : HD$. Hence $2nKk : B\beta + 2nKk :: 2nNH : HD$, and disjointly $2nKk : B\beta :: 2nNH : HD - 2nNH$, and $Kk : B\beta$, that is $AK : AB :: NH : HD - 2nNH$, and disjointly $AK : KB :: NH : HD - 2nNH - NH$, or $ND - 2nNH - 2NH$. Therefore $AK : \frac{1}{2n+2} KB :: NH : \frac{1}{2n+2} ND - NH$, and conjointly $AK + \frac{1}{2n+2} KB : \frac{1}{2n+2} KB :: \frac{1}{2n+2} ND : \frac{1}{2n+2} ND - NH$; that is, taking $KT = \frac{1}{2n+2} KB$, and $NV = \frac{1}{2n+2} ND$, we have $TA : TK :: VN : VH$; and $2n+2$ or $2 \times \overline{n+1}$ is twice the number of reflections of the given ray AB; because there is always one reflection more than the number of chords between the first and last points of reflection. *Q. E. D.*

410. *Corol.* 1. When AB is fixt in position VH is reciprocally as TA; and consequently V is the focus of rays that come parallel to AB, and T the focus of rays that come parallel to ED: and these principal focuses are found as above, by taking KT to KB as an unite to twice the number of successive reflections.

411. *Corol.* 2. The perpendicular subtenses Kk, Ll, Mm, &c. of the small angles at A, F, G, &c. are equal to one another. And the small arches Bb, Cc, Dd, &c. are in arithmetical progression as well as the perpendiculars $B\varrho$, $C\gamma$, $D\delta$, &c. because the little right angled triangles $Bb\varrho$, $Cc\gamma$, $Dd\delta$, &c. are equiangular.

PROPOSITION III.

Fig. 409 to 412. 412. *From the center* C *of a refracting circle* Bb, *betwixt given mediums, draw the perpendicular* CD *upon an incident ray* AB *and* CE *upon the refracted ray* BF *given in position; and if any point* A *be the focus of incident rays upon a small arch* B b, *and* F *be their focus after refraction, the ratio of the distances* BF *to* EF *will be compounded of the ratio of the similar distances* BA *to* DA, *of the ratio of the sines of incidence and refraction,* CD, CE, *taken directly, and of the ratio of the cosines,* BD, BE, *taken inversely.*

For

For let AbF be the nearest ray to ABF; and upon Ab draw the perpendiculars Cd, $B\delta$; and Ce, $B\eta$ upon bF; and the right angled triangles $B\delta b$, BDC will be similar, as appears by taking the common angle DBb from the right angles CBb, $DB\delta$. And for the same reason the right angled triangles $B\eta b$, BEC will be similar; and consequently the whole figures $B\eta\delta b$, $BEDC$ will also be similar. It is also to be observed that CD is to CE and Cd to Ce in the given ratio of the sine of incidence to the sine of refraction; and disjointly Dd is to Ee in the same ratio. For the same line CDd may be considered as perpendicular to both rays AB, Ab when the angle BAb is vanishing[a]. Hence because the triangles $BF\eta$, EFe are similar; and also $BA\delta$, DAd; the ratio of BF to EF or of $B\eta$ to Ee, which is compounded of the ratios $B\eta$ to $B\delta$, $B\delta$ to Dd, Dd to Ee, is also compounded of these ratios respectively the same as the former, BE to BD, BA to DA, CD to CE. *Q. E. D.* The position of the refracted ray BF, which is supposed to be given in this and the following propositions, will be determined hereafter by lemma 4.

a Art. 204.

413. *Corol.* 1. When the incident rays are parallel, the ratio of BF to EF is compounded of the direct ratio of the sines of incidence and refraction, and of the inverse ratio of the cosines; because the ratio of BA to DA is a ratio of equality when A is remote.

414. *Corol.* 2. The ratio of the perpendicular subtenses Dd, Ee, of the small angles at A and F, is invariable; being the same as of the sine of incidence CD to the sine of refraction CE.

LEMMA. I.

415. The ratio of the tangents, CT, CV, of any two angles, CBD, CBE, is compounded of the ratio of their sines, CD, CE, taken directly, and of their cosines, BD, BE, taken inversely. Fig. 413.

For the right angled triangles BCT, BDC are equiangular, and so are the right angled triangles BCV, BEC. Therefore the ratio of CT to CV, which is compounded of CT to CB and of CB to CV, or of CD to DB and of EB to EC, is the same as the ratio of the rectangle under CD, EB to the rectangle under DB, EC, which is compounded of the ratio of CD to CE and of EB to DB[b], that is of the sines directly and cosines inversely. *Q. E. D.*

b Euc. VI. 23.

PROPOSITION IV.

416. *Let* AB *and* Bf *be an incident and a refracted ray given in position. From* C *the center of curvity at* B, *or of the refracting circle, draw* CE *perpendicular to the refracted ray* Bf *produced; and let* Bf *be to* BE *as the tangent of the angle of incidence to the difference of the tangents of incidence and refraction; and let* Bf *be placed forward, that is according to the course of* Fig. 414 to 417.

of the refracted ray, if the surface of the denser medium be convex, otherwise backwards; and f *will be the focus of a slender pencil of rays that came parallel to* AB *upon the smallest arch at* B.

For the ratio of Bf to Ef, being compounded of the ratio of the sines of the angles of incidence and refraction directly and of their cosines inversely [a], is the same as the ratio of the tangents of those angles [b]; and disjointly Bf is to BE as the tangent of incidence to the difference of the tangents of incidence and refraction. The reason of the rule for the position of Bf is this, that the rays going forward in both cases, will converge in one case and diverge in the other. *Q. E. D.*

a Art. 413.
b Art. 415.

417. *Corol.* 1. Hence Bf is to Ef as the tangent of incidence to the tangent of refraction.

418. *Corol.* 2. Let a be the focus of rays that came parallel to fB; then we have Bf to Da as BE to BD, that is, the continuations of the cosines BE, BD to the focuses of rays that came parallel to them, are in the same ratio as the cosines themselves. For Bf is to Ef as the tangent of incidence to the tangent of refraction, and Ba to Da in the same ratio inverted. Therefore $Bf : Ef :: Da : Ba$, and disjointly $Bf : BE :: Da : DB$, and alternately $Bf : Da :: BE : BD$.

419. *Corol.* 3. The focus f may also be found by drawing lines in this manner. Draw CE perpendicular to the refracted ray Ef; EG perpendicular to the radius BC, and Gf parallel to the incident rays, and it will cut the refracted rays in their focus f. For let EG cut AB in H, and Gf being parallel to the base BH of the triangle BEH, we have $Bf : BE :: HG : HE$ [c], the same proportion as in the proposition: for GH and GE are tangents of the angles, GBH, GBE, of incidence and refraction.

c Euc. VI. 2.

420. *Corol.* 4. Hence it appears that while the angle of incidence continually increases, the focal distance Bf continually decreases, till it be reduced to nothing when the angle of refraction becomes a right one; or else till it be equal to the cosine of refraction when the angle of incidence is a right one. Consequently the focal distance is the longest when the angle of incidence is the least; and then this proposition degenerates to the second proposition of the third chapter [d]. For the tangents of very small angles are in the same ratio as their sines or arches [e].

d Art. 224.
e Art. 204.

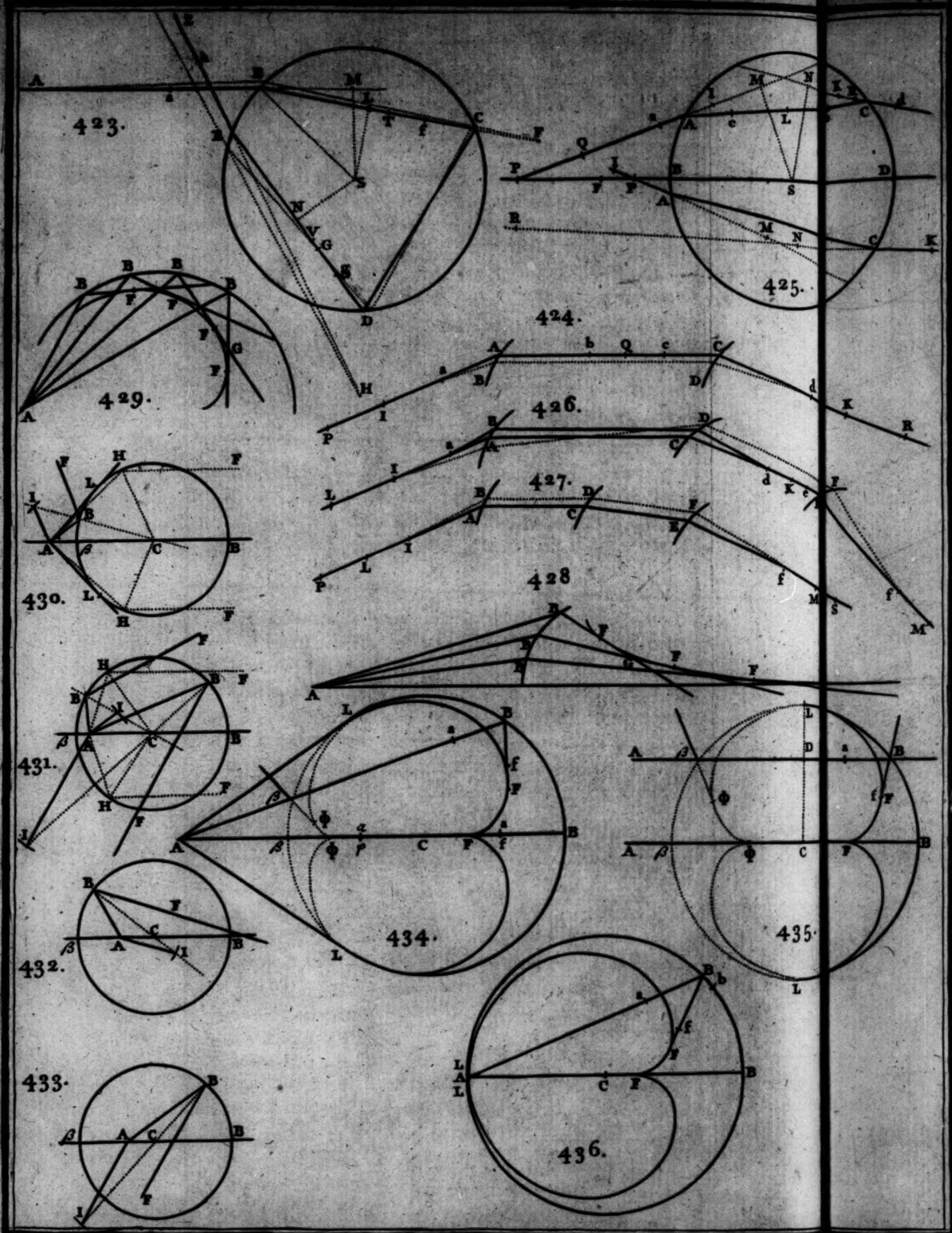
LEMMA II.

Fig. 418, 419.

421. The least increment of an angle of incidence, is to the contemporary increment of the angle of refraction, as the tangent of the angle of incidence, to the tangent of the angle of refraction.

Let two rays AB, aB, containing a very small angle ABa, be refracted at B along the lines BE, Be by a plane or by any curve-surface. From any point C, of the line BC perpendicular to that surface, draw CDd cutting the incident rays (produced) at right angles [f] in D and d; and like-

f Art. 204.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likewise CEe cutting the refracted rays (produced) at right angles in E and e. Then because CD is to CE and Cd to Ce in the same ratio of the sines, disjointly we have Dd to Ee as CD to CE. Now the ratio of the small angles (ABa or) DBd and EBe, which are the contemporary increments or decrements of the angles of incidence and refraction, being compounded of the ratio Dd to Ee and of BE to BD[a], that is of CD to CE and of BE to BD, is the same as the ratio of the tangents of incidence and refraction[b]. *Q. E. D.* a Art. 222. b Art. 415.

422. *Corol.* Hence if the angles of incidence and refraction of either of the rays ABe, aBe be invariable, while those of the other ray are varied a little, their small increments or decrements will be always to each other in an invariable ratio.

PROPOSITION V.

423. *Let a ray* AB *falling with any obliquity upon a refracting curve at* B, *be refracted along* BF *given in position; and let* Ba *be the focal distance of rays coming parallel to* FB; *and* Bf *the focal distance of other rays coming parallel to* AB; *then supposing any point* A *to be a given focus of incident rays, say as* Aa *to* aB *so* Bf *to* fF, *and place* fF *the same way with respect to* fB *as* aA *lyes with respect to* aB; *and* F *will be the focus of the refacted rays.* Fig. 414 to 417.

For let AGF be the nearest ray to ABF; and joining aG and fG, a ray aG after refraction at G will go along a line GH parallel to BF, by the supposition; and likewise a ray fG will be refracted through G along GI parallel to BA. But the angle AGa is to the angle FGH or GFf in a certain given ratio[c]; and in like manner the angle aAG or AGI is to fGF in the same given ratio[c]. Therefore alternately the angle AGa : ang. aAG :: ang. GFf : ang. fGF; and the sines of these small angles are in the same proportion[d]; and consequently in the triangles AGa, GFf, the sides opposite to those angles are also in the same proportion[e]; that is $Aa : aG :: Gf : fF$, or $Aa : aB :: Bf : fF$[f]. Let A approach towards a, and when it coincides with a, the line fF will become infinite; and therefore when A passes to the other side of a, the point F will also pass from an infinite distance to the other side of f. *Q. E. D.* Fig. 420. c Art. 422. d Art. 220. e Art. 221. f Art. 204.

424. *Corol.* 1. When the ray ABF is given in position the line fF is reciprocally as Aa.

425. *Corol.* 2. The same words applyed to the 421st figure are a demonstration of the same proportion for the focus of reflected rays. For now the angle AGa is to FGH or GFf in a ratio of equality[g], and so is the angle aAG or AGI to the angle fGF; and consequently the triangles AGa, GFf are equiangular; and therefore $Aa : aG$ or $aB :: Gf$ or $Bf : fF$. Which is another demonstration of the first of these propositions[h]; and the points a, f which are the focuses of parallel rays to FB and Fig. 421. g Art. 8. h Art. 402.

Fig. 422. and AB, may be found in this manner. Let two incident rays AB, ab describe two parallel chords of a circle $AabBcC$, and be reflected along
a Art. 9. 19. the chords BC, bc respectively equal to BA, ba[a], and let them cross in f. Then $2Bb = (Bb + Aa =$ arch $AB -$ arch $ab =$ arch $BC -$ arch $bc =) Cc - Bb$. Hence $Cc = 3Bb$ and $Bb : Cc :: 1 : 3$. Let the chords Ab, ab approach infinitely near to one another, and the figures Bfb, Cfc will be-
b Art. 204. come equiangular triangles[b], and consequently $Bf : fc$ or $fC :: (Bb : Cc ::) 1 : 3$; therefore $Bf = \frac{1}{4} BC = \frac{1}{4} BA$.

Proposition VI.

426. *Having the focus of a slender superficial pencil of rays falling with any obliquity upon a great circle of a sphere of any homogeneal matter, it is proposed to find the focus of the rays that emerge from the sphere by refraction, after any given number of successive reflections within it.*

Fig. 423. Let $ABCDEZ$ be the course of a ray incident at B, successively
c Art. 183 &c. reflected any given number of times, as at the points C, D[c], and emergent at E. And let A be the given focus of incident rays; F their focus after the first refraction at B; G their focus after the reflections at C and D; and H their focus after emergence by refraction at E. To find these focuses, let Bf and Ba be the focal distances of rays coming parallel to AB and FB respectively, and say as $Aa : aB :: Bf : fF$; which gives the focus F, by placing fF the same way with respect to Bf as Aa lyes
* Art. 423. with respect to aB*. Again from S the center of the sphere, draw SL, SN perpendiculars to the first and last chords BC, DE, and towards C and D the first and last points of reflection, take LT and NV severally in proportion to LC or ND, as an unite to twice the given number of successive reflections; then because F is the focus of the rays incident at C, say as $TF : TL :: TL$ or $VN : VG$, which gives the focus G after the last reflection at D, by placing VG the same way with respect to VN as TF lyes
* Art. 409. with respect to TL*. Lastly let Eg and Eb be the focal distances of rays coming parallel to ZE and GE respectively, then because G is the focus of the rays incident at E, say as $Gg : gE :: Eb : bH$; which gives H the focus of the emergent rays, by placing bH the same way with respect to
* Art. 423. Eb as Gg lyes with respect to gE*. *Q. E. J.*

427. *Carol.* 1. While the incident pencil revolves about the focus A with an angular motion in the plane of the circle $BCDE$, the proportio-
d Art. 420. nable lines Aa, aB, Bf, fF will all vary their lengths[d], and so will TL; and when TF, TL, Tf are continual proportionals the emergent rays will be parallel to EZ. For we had $Gg : gE :: Eb : bH$; therefore bH will be infinite when Gg is nothing, that is when VG equals Vg or Tf, which are always equal, by reason of the equal chords BC, DE and equal refractions at E and B. But by construction TF, TL, VG are always continual

nual proportionals; therefore when VG and Tf are equal, we have TF, TL, Tf continual proportionals.

428. *Corol.* 2. Hence if the incident rays be parallel, the emergent rays will also be parallel when Tf and TL become equal to one another; and consequently when Lf is to LC as an unite to the given number of successive reflections. For when the focus A is infinitely remote, the focus F will coincide with f, and consequently the continual proportionals TF, TL, Tf * will become equal to one another. * Art. 427.

429. *Corol.* 3. Hence drawing SM perpendicular to AB produced, and putting n for the given number of successive reflections, if the incident rays be parallel, the emergent rays will also be parallel when $BL : BM :: \overline{n+1}, SL : SM$. For in Corol. 2, we had $n : 1 :: LC$ or $LB : Lf$, and conjointly $n+1$ to 1 (as Bf to Lf, that is,) in a ratio compounded of SM to SL and of BL to BM *; therefore by compounding these ratios with the ratio of SL to SM, we have $\overline{n+1}, SL : SM :: BL : BM$. * Art. 413.

430. *Corol.* 4. Hence putting $I : R :: SM : SL$, and $m = n+1$, we shall have $BM : BS :: \sqrt{II-RR} : \sqrt{\overline{mm-1}, RR}$; which determines the angle of incidence BSM when the rays that came parallel to AB shall emerge parallel to EZ. And this is Sir *Isaac Newton*'s rule for determining the apparent diameters of the Rain-bows, as shall be explained hereafter. Opt. p. 148. For because $SM : SL :: I : R$, we have $SM \pm SL : SM :: I \pm R : I$. And in Corol. 3. we have $BL \pm BM : BM :: mR \pm I : I$. But $SM^2 + BM^2 = SL^2 + BL^2$ *, or $SM^2 - SL^2 = BL^2 - BM^2$, or $\overline{SM+SL} \times \overline{SM-SL} = \overline{BL+BM} \times \overline{BL-BM}$. * Euc. I. 47. Whence $SM + SL : BL + BM :: BL - BM : SM - SL$; that is, by substituting the values of these terms, which are given by the foregoing proportions, $\frac{I+R}{I} SM : \frac{mR+I}{I} BM :: \frac{mR-I}{I} BM : \frac{I-R}{I} SM$. Hence $\overline{I+R} \times \overline{I-R} \times SM^2 = \overline{mR+I} \times \overline{mR-I} \times BM^2$, and consequently $BM^2 : SM^2 :: II-RR : mmRR-II$; and conjointly $BM^2 : BS^2 :: II-RR : \overline{mm-1} \times RR$; and $BM : BS :: \sqrt{II-RR} : \sqrt{\overline{mm-1}, RR}$.

PROPOSITION VII.

431. *Having the positions of two given curves, betwixt three given mediums; and supposing the rays of a slender pencil in either of the outward mediums to be parallel, and to fall upon the curves with any obliquity; it is proposed to find their focus after refractions through both the curves.*

Let $aAbcCd$ be a ray refracted at A and C by the given curves AB, CD; in this given ray let a and d be the focuses of rays, which before Fig. 424. they

they emerged by refractions at AB and CD into the outward mediums, went both ways parallel and very near to AC in the inner medium. Also let b and c be the focuses of other rays, which before they emerged by refractions into the inner medium, came parallel to aA and dC respectively in the outward mediums. These focuses may be found by art. 416 and 401. Say then as $cb : bA :: Aa : aI$, and placing aI the same way with respect to aA as bc lyes with respect to bA, the point I will be the focus of rays after both refractions, which came parallel from without upon
* Art. 423. the curve CD*; because c is their focus after the first refraction. In like manner by saying as $bc : cC :: Cd : dK$, and by placing dK the same way with respect to dC as cb lyes with respect to cC, the point K will be the focus of other rays after both refractions which came parallel from with-
* Art. 423. out upon the other curve AB*. Q. E. J.

Fig. 425. 432. *Corol.* 1. Let the two surfaces AB, CD compose a sphere of homogeneal matter, and let S be its center. Draw OM perpendicular to the incident ray aA produced and ON perpendicular to the emergent ray dC produced. Then bisect aM and dN in I and K, and the points I, K will be the focuses of rays refracted through the sphere that came parallel
* Art. 418. to dC and aA. For bisecting AC in L, we have $Lc : LC :: Cd : CN$*, and conjointly $Lc : cC :: Cd : dN$; but by the proposition we have bc or $2Lc : cC :: Cd : dK$, and consequently $2Lc \times dK = (cC \times Cd =)\ Lc \times dN$ by the former proportion; therefore $dK = \frac{1}{2}\, dN$: and by the same argument $aI = \frac{1}{2}\, aM$.

433. *Corol.* 2. We had Cd to CN as the tangent of the lesser of the an-
a Art. 416. gles of incidence and refraction to the difference of their tangents[a]; and consequently the focus K will lye without or within the sphere, according as that lesser tangent is bigger or less than the difference of the said tangents; which difference increases to infinity, while the angles of incidence and refraction increase; that is while the ray AC goes farther and farther from the center of the sphere.

PROPOSITION VIII.

434. *The focus of incident rays being given, it is proposed to find their focus after both refractions through two given curves, betwixt three given mediums.*

Fig. 424. In the ray $IaACdK$ given in position, let a and d be the focuses of rays, which before they emerged by refractions at AB and CD into the outward mediums, went both ways parallel and very near to AC in the inner medium. Also let I and K be the focuses of other rays, which before their refractions through both the curves, came parallel to CK and AI in the outward mediums. Then let P be the focus of incident rays upon the surface AB; I and a the focuses of those rays which came the contrary way to the incident ones; and say as $PI : Ia :: dK : KR$, and placing

placing KR the ſame way with reſpect to Kd as IP lyes with reſpect to Ia, the point R will be their focus after both the refractions.

For by ſaying as $Pa : aA :: Ab : bQ$ and by placing bQ as uſual, the point Q will be their focus after the firſt refraction at AB*; the ſame point Q being the focus of incident rays upon CD, ſay again as $Qc : cC :: Cd : dR$, which being placed as uſual gives the focus R after both refractions[a]. Now by the firſt of theſe proportions and of thoſe in the foregoing propoſition, the rectangle $Pa \times bQ = (aA \times Ab =) bc \times aI$; and by the ſecond proportion in this and in the foregoing propoſition, the rectangle $Qc \times dR = (cC \times Cd =) bc \times dK$; then by reſolving the two former rectangles into the proportion of their ſides, it is as $Ia : Pa :: bQ : bc$, and disjointly (or conjointly) $PI : Pa :: (Qc : bc ::) dK : dR$ by reſolving the latter rectangles; and disjointly (or conjointly) $PI : Ia :: dK : KR$. Q. E. D. * Art. 423. a Art. 423.

435. *Corol.* 1. Hence when the ray $IACK$ is given in poſition, KR is reciprocally as PI; becauſe the rectangle $Ia \times dK$ is invariable.

436. *Corol.* 2. In the ſphere we have $PI : IM :: NK : KR$, which gives the focus R by placing KR the ſame way with reſpect to KN as IP lyes with reſpect to IM. Becauſe we had $IM = Ia$ and $KN = Kd$*. And this rule for the focus R is the very ſame as we had in the 236th article, when the rays paſs near the center of the ſphere. Fig. 425. * Art. 432.

437. *Corol.* 3. Draw PS cutting the circle in B and D; and let SF be the focal diſtance of the ſphere; then if SP be longer than SF, while the arch BA increaſes, the line KR will decreaſe to nothing, till all the points N, C, K, R coincide together in the ſurface of the ſphere. For while BA increaſes, AM decreaſes to nothing, being always half the chord of the arch cut off by PA produced. And IM decreaſes faſter than AM; being firſt bigger, then equal to and then leſs than AM*. But PI or $PA \mp AI$ increaſes; becauſe PA always increaſes, while AI firſt decreaſes to nothing, and then increaſes on the oppoſite ſide to AP. Therefore KR or $\frac{IM \times NK}{PI}$ decreaſes to nothing. * Art. 433.

438. *Corol.* 4. If SP be leſs than the focal diſtance SF, while the arch BA increaſes from nothing, the negative line KR will firſt increaſe to infinity and then decreaſe to nothing as before. For when A coincides with B, the points M, I coincide with S, F; and while BA increaſes, MI decreaſes[b] till PI becomes nothing, and KR infinite[c]; after this KR decreaſes to nothing as before. b Art. 437. c Art. 435.

439. *Corol.* 5. Hence it appears that NR firſt increaſes to infinity and then decreaſes to nothing.

440. *Corol.* 6. While NR is decreaſing the point R will enter the ſphere and return back again to its ſurface before NR vaniſhes at the ſurface. In the line PI take $QI : IM :: NK : KC$, and while AB increaſes, theſe

two ratios will at last become ratios of equality; (because the last ratio
a Art. 433. of NC to Cd will be infinite[a];) therefore QI will be less than PI. But
* Art. 436. $PI : IM :: NK : KR$*; and consequently $PI \times KR = QI \times KC$, and $PI : QI :: KC : KR$, and therefore KR is less than KC; and so the focus R will enter the sphere; and will return again to its surface, because NR vanishes there, when the incident and emergent rays are tangents to the sphere.

PROPOSITION IX.

441. *Having the position of three given curves, betwixt four given mediums, and supposing the rays of a slender pencil in either of the outward mediums to be parallel, and to fall with any obliquity upon the curves, it is proposed to find their focus after all the refractions.*

Fig. 426. Supposing the focuses a, d and I, K to be determined by the 8th proposition for two contiguous surfaces, as AB, CD; let the ray CeE be refracted at the surface EF along EfM; and let rays coming parallel to ME and CE have their focuses at e and f after refraction at EF only; whereof e being the focus of incident rays upon the surface CD, say as $eK : Kd :: aI : IL$, and placing IL as usual, the point L is the focus of the rays after three refractions that came parallel from without upon the
* Art. 434. curve EF*. Again say as $Ke : eE :: Ef : fM$, and placing fM as usual, the point M is the focus of other rays after three refractions, which came
* Art. 423. parallel from without upon the curve AB*. Q. E. J.

PROPOSITION X.

442. *Having the focus of a slender pencil of rays, incident with any obliquity upon any given number of curves betwixt given mediums, it is proposed to find the focus of the emergent rays.*

Fig. 427. Let I and f be the focuses of rays, which before they emerged by refractions into the two outward mediums, went both ways parallel and very near to the given ray in either of the inner mediums, suppose CE; and let L and M be the focuses of other rays, which before refractions through all the curves, came parallel to ME and LA respectively. Then let P be the given focus of the incident rays, L and I the focuses of those other rays which came the contrary way to the incident ones; and by saying as $PL : LI :: fM : MS$, and placing MS the same way with respect to Mf as LP lyes with respect to LI, the point S will be their focus after all the refractions.

This may be demonstrated by the 9th proposition in the very same manner as the 8th was demonstrated from the 7th, or in the very words of the 370th and 371st articles. And by the method of these demonstrations it appears that the rule here delivered for three surfaces serves universally for any other number. But to avoid being tedious I have omitted some

cases

cases of these propositions, as I did of those in the 8th chapter, such as are mentioned in the 399th article.

443. *Corol.* 1. Hence when the ray *PLACEMS* is given in position, *PL* is reciprocally as *MS*, because the rectangle under them is equal to the invariable rectangle under *LI*, *fM*.

444. *Corol.* 2. The same rules and demonstrations will serve also for finding the focus of rays successively reflected from any given number of curves, by quoting the first of these propositions instead of the fifth.

Definition of a Caustick

445. When an infinite number of incident rays, *AB*, *AB*, &c. that lye all in one plane of incidence, shall not belong to a single point or focus after their last reflections or refractions, but shall cut one another in an infinite number of points; if a curve *FFF* be supposed of such a shape as to touch every one of the reflected or refracted rays *BF*, *BF*, &c. produced if need be, in the points *F*, *F*, &c. the curve *FFF* is called *a Caustick by reflection or by refraction*, according as the name is applied to reflected or refracted rays. Fig. 428, 429.

446. *Corol.* 1. Let any two tangents *BF*, *BF* intersect one another in *G*; and supposing these tangents to approach to one another till they coincide, the points of contact and the point of intersection will also approach to one another till they also coincide. It is manifest therefore that a reflected or refracted ray touches the caustick in that point of the ray, where its intersection with the next ray vanished, when they were supposed to coincide.

447. *Corol.* 2. Therefore conceiving two incident rays infinitely near to each other to revolve about their focus *A*, in the plane of incidence, the focus *F* or intersection of the reflected or refracted rays will describe the caustick above defined. Which is called a real or an imaginary caustick according as *F* is the focus of converging or diverging rays. To get an idea of the shapes of causticks formed by rays reflected from a circle or any curve, it is requisite to consider the different inclinations of the reflected rays to that diameter of the circle which passes through the focus of incident rays, or to the axis of the curve.

Lemma III.

448. Let *A* be the focus of incident rays upon a circle βBH whose center is *C*. In the angle of incidence *ABC*, or in its complement to two right angles, inscribe a line *AI* equal to the incident ray *AB*; and the reflected ray *BF* will be parallel to *AI*. Fig. 430 to 433.

For the angles *ABI*, *AIB*, *IBF* are equal to one another[a]. *Q. E. D.* a Euc. I. 5, 29

449. *Corol.* 1. Draw the axis *AC*, cutting the circle at right angles in β and *B*; and let the lines *AL*, *AL* touch it in *L* and *L*; and the points

L, L will divide the circumference into two arches whereof LBL, the remoter from A, will be concave towards A, and $L\beta L$ the nearer to A will be convex towards it. And when the point B comes to L the point I will alſo coincide with it, and ſo the reflected ray will proceed in the direction of the incident ray.

450. *Corol.* 2. In the reflecting circle inſcribe two lines AH, AH ſeverally equal to AC, the diſtance of the focus from the center; and the rays that are reflected from the remoter arch HBH will all converge towards the axis AC; and thoſe which are reflected from the nearer arch $H\beta H$ will all diverge from AC; and the two rays, HF, HF, that are reflected from the points H, H, being parallel to the axis AC, will divide the converging rays from the diverging ones. For ſuppoſing the ray AB to approach to AH and to coincide with it, the line AI will approach to AC and coincide with it; and then the reflected ray BF, which is always parallel to AI, will now become parallel to AC. But while AB is ſhorter than AH or AC, the equal line AI will alſo be ſhorter than AC, and conſequently will be ſituated on the ſame ſide of the axis AC as AB, and therefore the reflected ray BF, being parallel to AI, will diverge from the axis AC. On the other hand while AB is longer than AH or AC, the line AI will alſo be longer than AC, and conſequently will be ſituated on the contrary ſide of the axis, AC, to AB; and therefore BF being parallel to AI, will now converge toward the axis AC.

451. *Corol*, 3. Hence it is evident, that when the focus A is nearer to the center of the circle than half a ſemidiameter, the rays reflected from the whole circle will all converge towards the axis AC.

Proposition XI.

Concerning the ſhapes and properties of cauſticks by reflection.

Fig. 434. 452. *Caſe* 1. Let A be the focus of incident rays, and be out of the reflecting circle $LBL\beta$; and let the lines AL, AL, touch it in L and L; and while the point of incidence B of the ray AB is deſcribing the arch $LBBL$ which is concave towards the focus A, the conjugate focus F of an infinitely ſlender pencil, will deſcribe a real cauſtick $LFFL$*; and while the point of incidence β is deſcribing the arch $L\beta\beta L$ which is convex towards the focus A, the conjugate focus Φ will deſcribe an imaginary cauſtick $L\Phi\Phi L$*. The real and imaginary cauſticks are both contained within the reflecting circle; and each of them conſiſts of two ſimilar and equal parts on oppoſite ſides of the axis AC, where each couple of ſimilar parts unite, and form two cuſps or ſharp points at F and Φ on oppoſite ſides of the center C. All this appears by the 402d article, by which $Bf = Ba = \frac{1}{4} B\beta$ and $Aa : aB :: Bf : fF$, and therefore the cauſtick's tangent BF increaſes continually while AB and $A\beta$ move from AL towards the axis AC.

* Art. 447.

* Art. 447.

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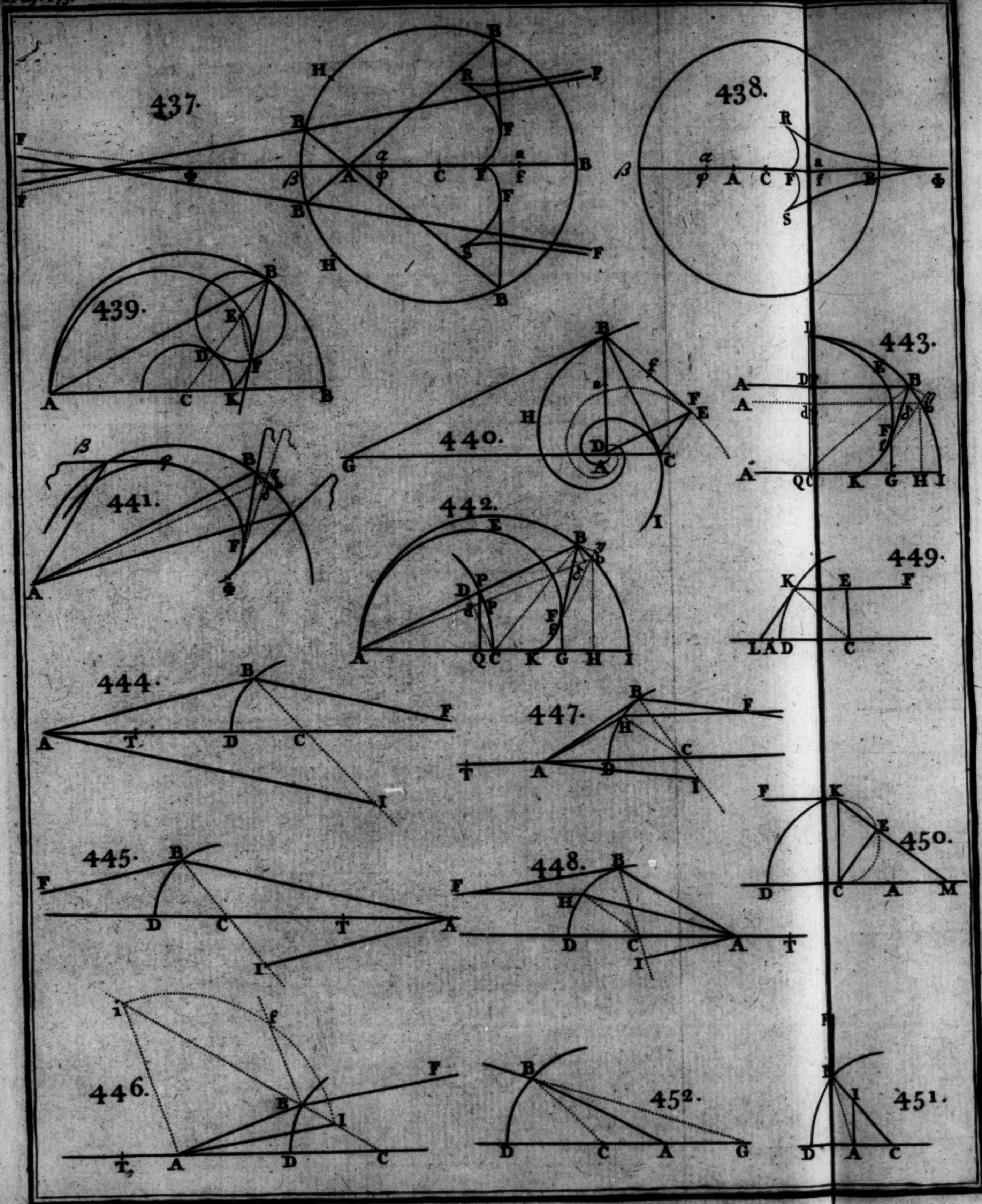
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453. *Case* 2. Let the focus *A* be removed to an infinite distance; and the caustick's tangent *BF* will every where be equal to *Bf* or *Ba* or one quarter of the incident chord *Bβ*, supposed to move parallel to the axis *AC*. Fig. 435.

454. *Case* 3. Let the focus *A* be in the circumference of the reflecting circle, and the caustick's tangent *BF* will be every where equal to one third of the reflected or incident chord *AB* *. For *Bf* or *Ba* being equal to $\frac{1}{4}AB$, we have *Aa* to *aB* as *Bf* to *fF* as 3 to 1; and therefore $fF = \frac{1}{12}AB$ which being added to *Bf* or $\frac{3}{12}AB$ gives $BF = \frac{1}{3}AB$. The imaginary caustick is here vanished and the parts of the real one come quite round to the focus *A* and there touch the circle. Fig. 436. * Art. 402.

455. *Case* 4. Let the focus *A* enter the reflecting circle, but not so far as $\frac{1}{4}$ of its diameter *βB*; and the rays reflected from the nearest point *β* will diverge from an imaginary cusp Φ in this diameter produced backwards[a]. Besides this cusp at Φ and the correspondent cusp at *F*, formed by reflections from the other end *B* of the said diameter, there are two other cusps *R* and *S* on opposite sides of this diameter. These three cusps belong to the real caustick, which has also two legs *RF*, *SF* infinitely extended, that approach to two asymptotes *BF*, *BF*; to which there also belong two other legs Φ*F*, Φ*F* of an imaginary caustick approaching to their contrary sides, and extended contrary ways to the former. For let the point of incidence *B* revolve round the circle beginning from the nearest point *β*; and so long as the incident ray *AB* continues less than $\frac{1}{4}$ of the incident chord *BAB*, the reflected rays will diverge from the imaginary caustick Φ*F* *; and when *AB* equals $\frac{1}{4}$ of its chord, the reflected ray becomes an asymptote *BF*, or a tangent to the curve at an infinite distance[b]; and consequently when *AB* is bigger than $\frac{1}{4}$ of its chord, the reflected rays will converge towards one another and form an opposite leg *RF* by the motion of their focus *F*; which first approaches towards the point of incidence *B* till it comes to a certain limit, and then recedes from it, as the incident chord grows still longer; so that by means of these contrary motions of *F* in the reflected ray *BF*, and of the gradual change of its inclination, a cusp or point of retrogression will be formed at *R*; after the manner that the cusp or point of retrogression is formed in the axis at *F* by the lengthening and shortening of *BF*, and by the continual change of its inclination. Fig. 437. a Art. 207. * Art. 402. b Art. 405.

456. *Case* 5. Let the focus *A* bisect the semidiameter *βC*, and the cusp Φ being now removed to an infinite distance[c], the asymptotes *BF*, *BF* will coincide with the axis; and when *A* advances nearer to the center, the two legs *RF*, *SF* will meet at a finite distance *A*Φ on the contrary side of *A*; and so this caustick will have 4 real cusps; and when *A* comes into the center it will be contracted into this single point. Fig. 437, 438. c Art. 405.

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457. *Case* 6. Laſtly if the incident rays fall upon the oppoſite ſide of the circle converging towards *A*, all theſe cauſticks will be the very ſame as before, except that the real and imaginary parts will change places.

Fig. 439. 458. Some of theſe cauſticks may be deſcribed after the manner of deſcribing a cycloid. For example, in the third caſe where the focus *A* was in the circumference at *A*, and *BF* the tangent of the cauſtick was $\frac{2}{3}$ of the chord *AB*, of the reflecting circle; divide any ſemidiameter *BC* into 3 equal parts *CD*, *DE*, *EB*, and with the center *C* and ſemidiameter *CD* deſcribe the circle *DK*, cutting *AC* produced in *K*; alſo with the center *E* and ſemidiameter *ED* or *EB* deſcribe the circle *BFD*, cutting the reflected ray *BF* in *F*; then if the latter circle *BFD* be rolled, like a wheel, without ſliding upon the convexity of the former circle *DK*, the given point *F*, of the moveable circle, will deſcribe the cauſtick *AFK*. For joining *EF*, ſince the equicrural triangles *BEF*, *BCA* are equiangular, and ſince *BE* is $\frac{1}{3}$ *BC*, the line *BF* is therefore $\frac{1}{3}$ of *BA*; and conſequently the point *F* is in the cauſtick[a]. And ſince the angles *DEF*, *DCK*, which are the complements of the equal angles *BEF*, *BCA*, are equal, the arches *DF*, *DK*, deſcribed with equal ſemidiameters are alſo equal.

a Art. 454.

Fig. 440. 459. The cauſtick of any curve, whoſe radius of curvity at every point is known, may alſo be determined by art. 402. For inſtance, let *AHB* be an equiangular ſpiral ſurrounding its pole *A*; whoſe eſſential property is this, that while the revolving ray *AB* increaſes or decreaſes, the magnitude of the angle *ABG*, made by the ray *AG* and the curve at *B* or its tangent *BG*, is invariable. Let the line *GAC* be always at right angles to *AB* and let it meet *BC*, perpendicular to *BG* in *C*; and in like manner as the point *B* of the moveable angle *ABG* deſcribes the ſpiral *BH*, ſo the point *C* of the equal angle *ACB* will deſcribe another equiangular ſpiral *ACI* about the pole *A*. From hence it appears that *C* is the center and *CB* the ſemidiameter of a circle of equal curvity to the firſt ſpiral at *B*, becauſe it is perpendicular to it at *B* and a tangent to the other ſpiral at *C*.

Now ſuppoſing *A* the focus of incident rays upon the ſpiral *AHB*, the focus *F* of the reflected rays, as *BF*, will deſcribe a third equiangular ſpiral *AF*, differing from either of the former in poſition only. For, from the center *C* of a circle of equal curvity to the ſpiral at *B*, let down the ſines of incidence and reflection *CD*, *CE* upon *AB*, *BF*; and ſince *D* always falls upon the focus *A*, the conjugate focus *F* will always fall upon *E* *. And becauſe *AB*, *BE* are equal, and equally inclined to the perpendicular *BC* and conſequently to the tangent *BG*, the line *AF* will be parallel to *BG*; and ſo the angle *AFB*, made by *AF* and *BF*, the tangent of the cauſtick at *F*, will be equal to the invariable angle *ABG*. *Q. E. D.*

* Art. 402.

Fig. 441. 460. The length of any portion of a cauſtick formed by any reflecting curve, is equal to the ſum of the incident and of the reflected ray which termi-

terminates one extremity of that portion, diminished by the sum of the incident and of the reflected ray which terminates its other extremity.

Imagine the tangent BF to be a flexible line or string, which being extended both ways does lap or unlap it self upon the convexity of the caustick without sliding, so as to measure the length of any portion of it to which it is applied. And having made the same construction as in prop. 1, since the triangles $Bb\delta$, $Bb\eta$ were there shewn to be equal, the increment $b\delta$ of the incident ray BA, is every where equal to the decrement $b\eta$ of the string $BF\Phi$, reckoning from any fixt point Φ. And if the point B moves the contrary way, the decrements of AB are every where equal to the increments of the string. Taking therefore the correspondent sums of these increments or decrements, it follows, when AB, BF are in any other place as $A\beta$, $\beta\Phi$, that when AB increases, $A\beta - AB = \Phi F + FB - \Phi\beta$; whence by taking equal things from equal things, $A\beta + \beta\Phi - AB - BF = F\Phi$, the portion of the caustick: and when AB decreases, $AB - A\beta = F\varphi + \varphi\beta - FB$; whence $AB + BF - A\beta - \beta\Phi = F\varphi$.

461. In the third case when A is in the circumference, it follows from the rule above that the length of the portion $AF = AB + BF = \frac{4}{3}AB$: and in the 2d case, when the incident rays are parallel, the portion $LF = DB + BF = \frac{3}{2}DB$, the line DB being half $B\beta$. Fig. 436. Fig. 435.

462. The density of the rays in any particle of a caustick may be determined in this manner. Let the incident rays AB, Ab be reflected from a small arch Bb of any curve BI whose axis is AI; and let the reflected rays touch the caustick FfK in F and f. With the center A and with any given semidiameter AC describe an arch CpP cutting the incident rays AB, Ab in P and p, and the density of the rays in the small arch Ff, will be to the uniform density of the same rays in the arch Pp as Pp to Ff. For supposing all the rays incident upon the arch Bb to be regularly reflected, the same number of rays will exist every moment in the lines Pp, Ff; and consequently their densities in these lines are reciprocally as their lengths. Hence if the magnitude of the arch Pp be supposed invariable, the density of the rays in the particle Ff will be reciprocally as its length. Fig. 442.

463. Draw the lines PQ, FG perpendicular to the axis AI, and when the whole figure is turned about this axis, all the rays that flow from A and are reflected from the surface described by the curve BI, will touch a superficial caustick described by the circular motion of the linear caustick $EFfK$; and the density of the rays in any part of this caustick described by a small arch Ff, will be to the uniform density of the incident rays in the spherical surface described by the arch CPp, as the rectangle under Pp and PQ to the rectangle under Ff and FG. For the same number of rays exists every moment in the rings described by the circular motion of the lines Pp, Ff; and their densities, being uniform in each ring, are reciprocally as the magnitudes of the rings. But the ring described by

Pp is equal to the rectangle under Pp and the periphery described by the point P, and the ring described by Ff is equal to the rectangle under Ff and the periphery described by the point F; and since the ratio of these peripheries is the same as the ratio of their semidiameters, the former rectangle is to the latter as $Pp \times PQ$ to $Ff \times FG$.

464. To give an instance or two of this latter rule, let the reflecting surface ABI be spherical and let the focus A be in this surface whose center is C; and let the semidiameter AP equal AC. And the density of the rays in the superficial caustick at F, will be to the uniform density of the incident rays in the spherical surface CP, as the semidiameter AC to $\frac{8}{3}$ of the ordinate FG. For the length of the portion AEF, of the linear caustick, is equal to $\frac{4}{3} AB$* and consequently the least increment or decrement of this caustick is equal to $\frac{4}{3}$ of the increment or decrement of AB, that is $Ff = \frac{4}{3} b\beta$. Draw CD perpendicular to AP and we have CD equal to PQ, both being sines of the same arch CP. And because the triangles PpA, $B\beta A$, and also $B\beta b$, BCD are similar, the ratio of Pp to Ff, being compounded of Pp to $B\beta$, $B\beta$ to $b\beta$ and $b\beta$ to Ff, is compounded of AP to AB or $2BD$, BD to CD and 3 to 4, which make the ratio of $3AP$ to $8CD$ or $8PQ$. Therefore by the rule the density at F is to the density at P or C (as $Pp \times PQ$ to $Ff \times FG$) as $3AP \times PQ$ to $8PQ \times FG$, or as AC to $\frac{8}{3} FG$.

* Art. 461.

465. Hence drawing BH perpendicular to the axis ACI, the density at F in the superficial caustick, is as its ordinate FG, or as the rectangle under BH and HI. For I find that FG is to BH as HI to $\frac{1}{2} IC$, which is not worth the trouble of a demonstration. Hence it appears that the density of the rays in the axis at K and A, is infinitely greater than at any finite distance from it.

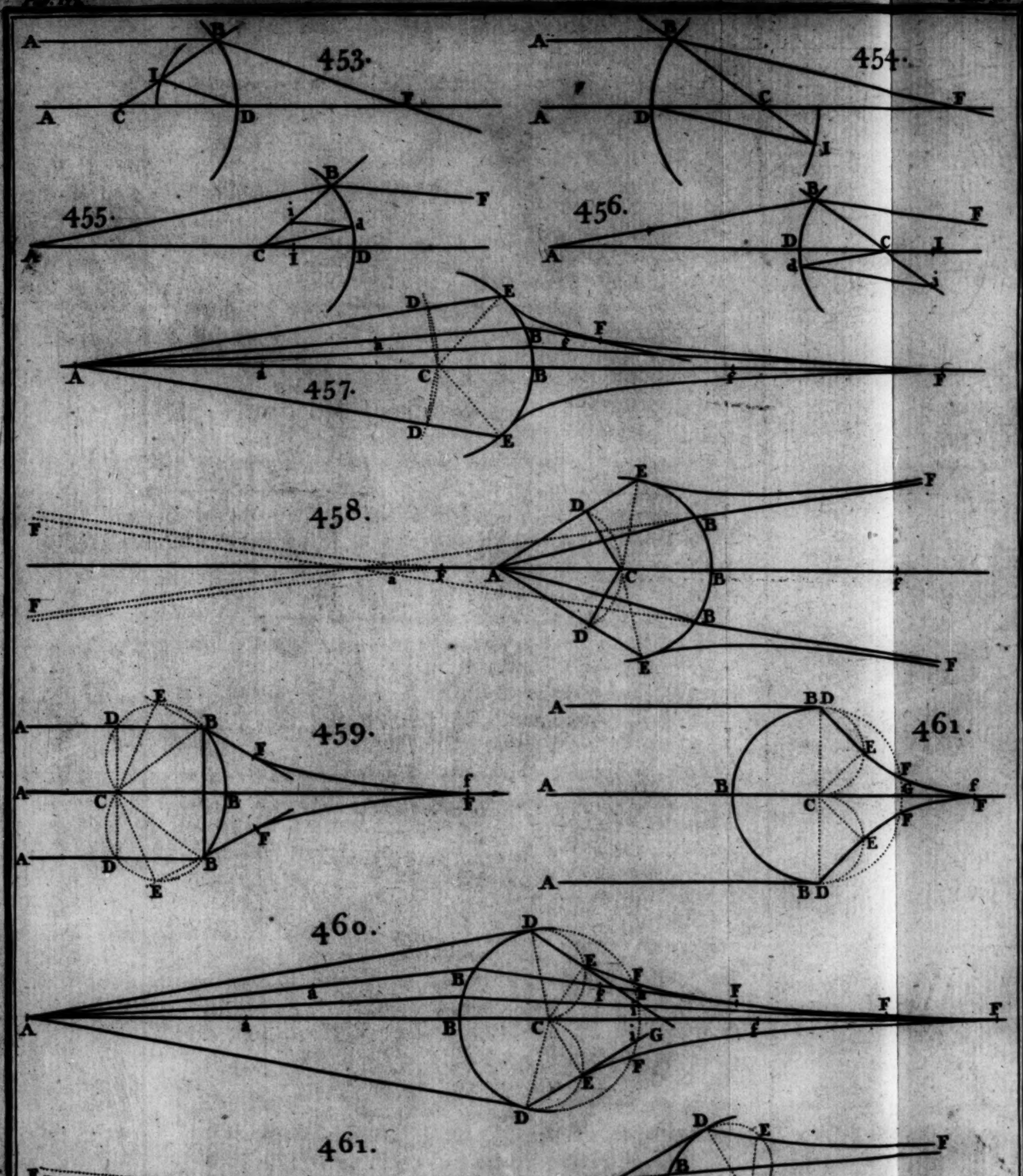
Fig. 443. 466. When the focus A is at an infinite distance from the reflecting spherical surface LBI, the density of the rays at any point F of the superficial caustick, described by the revolution of the linear caustick LFK about the axis ACI, is to the uniform density of the incident rays upon a perpendicular plane CDL, as BD to FG, that is as the cosine of the angle of incidence to the ordinate drawn from the point F. For the portion LEF of the linear caustick is equal to $\frac{1}{2} BD$*; and therefore $Ff = \frac{1}{2} b\beta$. But Pp is to Ff in a ratio compounded of Pp or $B\beta$ to $b\beta$ and of $b\beta$ to Ff, that is of BD to DC and of 2 to 3. Therefore the density at F is to the density at D as ($Pp \times PQ$ to $Ff \times FG$* that is as) $2BD \times PQ$ to $3CD \times FG$, or as BD to $\frac{1}{2} FG$.

* Art. 461.

* Art. 463.

467. Hence the density at F is as BD directly and FG inversely; or as BD directly and CD cube inversely. For I find that FG is to CD as CD^2 to CI^2. Hence it appears that the density of the rays at the cusp K is infinitely greater than when the ordinate FG has a finite magnitude.

468.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468. By these rules we have the proportions of the heat or warmth of the rays in the several parts of these caustics, both with respect to one another and to that of the incident rays upon a perpendicular surface; upon supposition that the heat of the rays in any surface is proportionable to their density, whatever be their mutual inclinations to one another.

LEMMA. IV.

469. In the angle of incidence *ABC* or its complement to two right ones, inscribe a line *AI* taken in proportion to *AB* as the sine of incidence to the sine of refraction; and the refracted ray *BF* will be parallel to *AI*. Fig. 444, 445.

For in the triangle *ABI*, the sine of the angle *ABI* is to the sine of the angle *AIB* as *AI* to *AB* *, that is by construction as the sine of incidence to the sine of refraction. But *ABI* is the angle of incidence or its complement to two right ones, and therefore *AIB* or *IBF* is the angle of refraction or its complement to two right angles.

* Art. 221.

It must be observed that a circle whose center is *A* and semidiameter is *AI*, will cut *BC* produced in two points *I* and *i*, and consequently two lines *BF*, *Bf* may be drawn from *B* respectively parallel to *AI* and *Ai*, making equal angles with *CBi* on each side of *B*; but it is easy to distinguish which of the lines *BF*, *Bf* is described by the refracted ray, by observing whether the refraction be made towards the perpendicular *BC* or from it. Fig. 446.

470. *Corol.* 1. Hence when the surface of the denser medium is convex in the axis *AC* take *CT* to *TD* as the sine of incidence to the sine of refraction; and if *CA* be greater than *CT*, all the rays falling on the circle *DB* will converge towards the diameter *CD*. For then the ratio of *CA* to *AB* will be always nearer to a ratio of equality than the ratio of *CT* to *TD* or of *IA* to *AB* by construction; and consequently *IA* and *AB* will always be on contrary sides of the axis *AC*, and therefore *BF* will always converge towards it; and will cut it in greater angles while *DB* grows greater. Fig. 444, 445.

471. *Corol.* 2. But if *CA* be less than *CT*, let the incident ray *AH* be to *AC* as the sine of refraction to the sine of incidence, and the refracted ray *HF* will be parallel to the axis; and all the rays whose points of incidence are farther from the axis than *H* will converge towards the axis and the rest that are nearer will diverge from it. For in the triangle *ACH* the sine of the angle *AHC* is to the sine of the angle *ACH* or of *CHF* as *AC* to *AH* *, or in the ratio of refraction; therefore *CHF* is the angle of refraction. Now if *AB* be farther from the axis than *AH*, then *AI* and *AB* must be on contrary sides of the axis, to be in the same ratio as *AC* to *AH*; and therefore *BF* being parallel to *AI* converges towards the axis. Fig. 447, 448.

* Art. 221.

But when AB comes between AH and AC, then AI must do so too, and so BF will diverge from the axis.

Fig. 449. 472. *Corol.* 3. When the incident rays enter a convex surface of a denser medium, erect CE perpendicular to the axis CD, and let CE be to the semidiameter CD or CK, as the sine of refraction to the sine of incidence; then draw EK parallel to the axis, and let KL touch the circle in K and cut the axis in L; and if CA be less than CL, all the rays which came from A will diverge from the axis after refraction. Because the tangent LK will be refracted into KE. But when the rays go from the denser medium into the rarer, let CK, erected perpendicular to the axis, be
Fig. 450. the diameter of a semicircle CEK, in which inscribe CE in proportion to CK as the sine of incidence to the sine of refraction; and drawing KE cutting the axis in M, if CA be less than CM, all the refracted rays will diverge from the axis. For MK will be refracted into the tangent KF.
Fig. 451. The rest appears from the 2d corol. If in this last case A advances first to the center and then still nearer to the surface, the rays flowing from the center will emerge unrefracted, and then will diverge from the axis the contrary way.

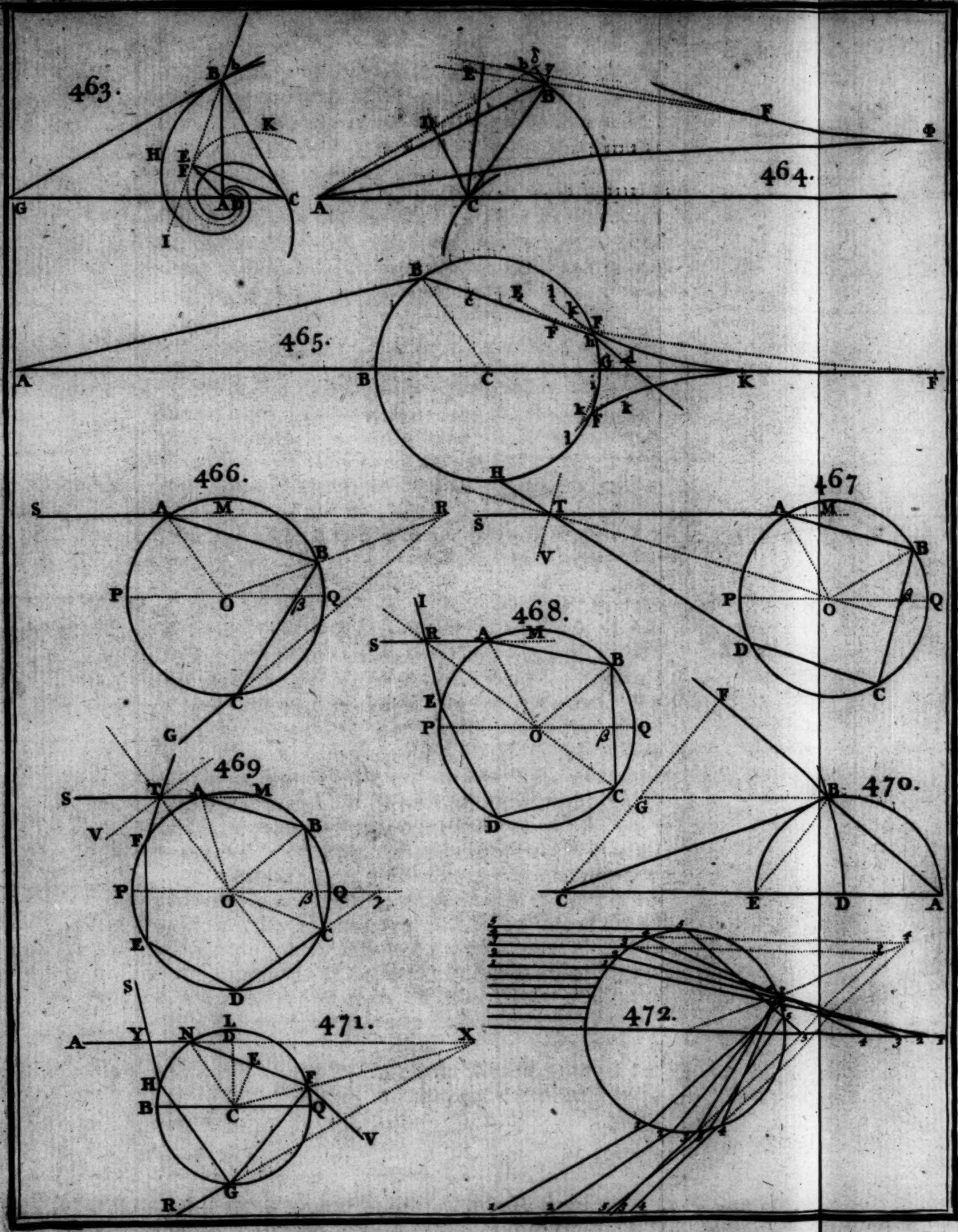
Fig. 452. 473. *Corol.* 4. It is observable, that taking CA, CB, CG continual proportionals in the ratio of the sine of incidence to the sine of refraction, and placing A and G both on the same side of C in the denser medium, all the refracted rays will diverge accurately from the given point G. For the triangles CAB, CBG are equiangular, having their sides about the com-
a Euc. VI. 8. mon angle C proportionable[a]: and so the sine of the angle of incidence CBA, is to the sine of the angle CBG or CAB, as the opposite side CA
* Art. 221. to the opposite side CB*, that is by construction in the ratio of the sines that measure the refraction; and consequently CBG is the angle of refraction; and the points A, G are invariable.

Fig. 453, 454. 474. *Corol.* 5. If the incident rays as AB come parallel to CD; in the angle BCD, or its complement to two right ones, inscribe a line DI taken in proportion to DC as the sine of incidence to the sine of refraction; and the refracted ray BF will be parallel to DI. For in the triangle DCI the
* Art. 221. sine of the angle DCI, is to the sine of DIC, as DI to DC*, that is by construction as the sine of incidence to the sine of refraction. But DCI is equal to the angle of incidence ABI or its complement to two right ones; and therefore DIC or FBC is the angle of refraction or its complement to two right ones.

475. *Corol.* 6. Hence we have a practical method of drawing any number of refracted rays very expeditiously, by describing an arch with the center D and given semidiameter DI; by drawing any line CB cutting this arch in I; by joining DI and drawing BF parallel to it.

476. *Corol.* 7. Hence while the arch DB increases, the line CF decreases. For the triangles CFB, CDI being equiangular, we have $CF : CB :: CD : CI$. Therefore CF is reciprocally as CI.

477.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477. *Corol.* 8. When the incident rays diverge from a focus A, the refracted rays may be drawn with the like expedition in this manner. Take a line DI to DC as the sine of incidence to the sine of refraction; and draw a semidiameter Cd parallel to the incident ray AB; and in the angle dCB, or in its complement to two right ones, inscribe a line di taken equal to the constant line DI; and draw the refracted ray parallel to di. For in the triangle dCi, the sine of the angle dCi is to the sine of diC, as di to dC, as the sine of incidence to the sine of refraction, by construction. But the angle dCi is equal to the angle of incidence ABC or to its complement, and therefore diC or FBC is the angle of refraction or its complement. Fig. 455, 456.

PROPOSITION XII.

478. *Concerning the shapes and properties of caussticks by refraction.*

Having determined the position of any refracted ray, by the foregoing lemma and its corollaries, and also the point in that ray where the nearest ray will intersect it, by art. 423; the points of a caustick are thereby determined [a]. But to form an idea of their shapes it is necessary to consider several cases. [a] Art. 446.

479. *Case* 1. Let the focus A and the center C of a refracting circle EBE be both in the denser medium. Upon the diameter AC having drawn a circle $ADCD$, inscribe in it the equal chords CD, CD, taken severally in proportion to the whole sine CB or CE, as the sine of incidence to the sine of refraction; and drawing the incident rays ADE, ADE, the legs of the caustick will begin from E and E*, where they touch the refracting circle, and will approach towards each side of the axis ACF till they meet it in the principal focus F, and there form a cusp or sharp point; provided A be farther from the center than a the focus of rays that come the contrary way parallel to CA. But if A advances to a, the distance fF will be infinite and so the axis ACF will become an asymptote to the legs of the caustick. And if A advances nearer than a, the legs will open and have two asymptotes BF, BF to which the rays emerging from B will run parallel. This must necessarily happen in a certain position of BA, that is when rays coming the contrary way parallel to FB, are collected to A. For the focal distance of parallel rays decreases as the point of incidence recedes from the axis till it equals ED*. There are also two other imaginary legs belonging to the same asymptotes, that extend themselves from the focus F, now situated on the other side of the center. Fig. 457, 458. * Art. 420. 423. Fig. 458. * Art. 420.

480. *Case* 2. When CA is to CB as the sine of incidence to the sine of refraction the caustick will be contracted into a point F from which all the rays will diverge by art. 473.

481. *Case* 3. The 459th figure represents a piece of a caustick made by a thick plano-convex lens BBB when a pencil of parallel rays falls per-

 pendicu-

pendicularly upon its plane surface; and consequently are only refracted by the spherical surface; the position of the refracted rays that fell upon the circumference of the lens is determined as in the next case.

Fig. 460, 461, 462. 482. *Case* 4. While *C* remains in the denser medium, transpose the focus *A* into the rarer, and draw the tangents *AD*, *AD* to the refracting circle *DBD* and joyn *CD*, *CD* upon which as diameters describe the semicircles *CED*, *CED* towards the denser medium, and inscribe in them the lines *CE*, *CE*, taken in proportion to the whole sine *CD*, as the sine of refraction to the sine of incidence. And the legs of the caustick beginning from *E** in the direction *DE*, *DE* will approach towards the axis *AC*, till they meet in the principal focus *F*; provided *CA* be greater than *Ca*; or will have the same positions as in the first case.

* Art. 420. 423.

Fig. 460. 483. The focus *A* being in the rarer medium, let the circle *DBD* be continued quite round, and let it cut the caustick in *F* and *F*, the axis *AC* in *G* and any other ray *ABF* in *b*; and while the ray *AB* is carried with an angular motion round about *A* from the axis *AC* towards the tangent *AD*, the arch *Gb* will first increase till it equals the arch *GF*, and then will decrease again till it equals the arch *Gi*, cut off by the last refracted ray *DEi*. This is manifest from the motion of the refracted ray *BbF* while it touches the convexity of the caustick in *F*, provided the focus *A* be so remote from the surface that the last refracted ray *DEi* may converge towards the axis *AC*.

Fig. 463. 484. Causticks made by refractions at other curves are also determinable by the 423d article. For instance, the imaginary caustick *AFK* made by refractions at an equiangular spiral *AHB* is also an equiangular spiral, their common pole *A* being the focus of incident rays. For supposing what has already been said of this curve [a]; from the center *C* of a circle of equal curvity to it at *B*, let down the sines of incidence and refraction *CD*, *CE* upon the incident and refracted rays *AB* and *BFI* produced backwards. And because *D* coincides with the focus *A* of incident rays, *E* does also coincide with *F* the focus of refracted ones [b]. Joyn *AE* or *AF*, and because the angular points *A* and *E* of the right angles *CAB*, *CEB* are in a semicircumference whose diameter is *CB**, the angles *CBA*, *CEA* standing upon the same chord *CA* are equal [c] and being substracted from the right angles *CBG*, *CEI*, the remaining angles *ABG*, *AEI*, which the lines *AB*, *AE* or *AF* do make with the curves, are every where equal; which is the property of the spiral. So that this caustick-spiral differs from the other in position only.

a Art. 459.
b Art. 418, 423.
* Euc. III. 35.
c Euc. VI. 23.

Fig. 464. 485. To find the length of any caustick by refractions, imagine the reflected ray *BF* produced, to be unlaped like a string from the convexity of the caustick $F\varphi$; then because the figures $b\delta\eta B$, *CDEB* are similar [d], the increment $b\delta$ of the incident ray *AB*, is every where to the decrement $b\eta$ of the string $BF\varphi$, in the given ratio of the sine *CD* to the sine *CE*;

d Art. 421 demonstr.

(for

(for which put n to m;) and therefore when AB, BF come into a new position $A\beta$, $\beta\varphi$, the sum of the increments of AB, that is $A\beta - AB$, is to the sum of the decrements of the string, that is $BF + F\varphi - \beta\varphi$, as n to m. Whence $\frac{m}{n} \times \overline{A\beta - AB} = BF + F\varphi - \beta\varphi$, or $\frac{m}{n} \times \overline{A\beta - AB} + \beta\varphi - BF = F\varphi$.

486. To find the points of any caustick made by two successive refractions, let the ray BFb which touches the caustick EFF in F, (made as before by the first refraction of the rays), meet with another refracting curve GbF, or with the same curve continued; and let it be refracted at b into the line bd; in which let bd be the focal distance of other rays coming parallel to Bb, and in bB let bc be the focal distance of other rays coming parallel to db; then since F is the focus of incident rays upon the curve bG, say as $Fc : cb :: bd : dk$, and placing dk as usual[a], the point k will be the focus of a slender pencil after both refractions, or a point of the second caustick KFk; whose points may also be found by art. 434 and 436, without finding the points F of the first caustick. Fig. 465.

a Art. 423.

487. Hence it will appear that a caustick made by refractions through a circular section of a cylinder or a great circle of a sphere, will have such a shape as is here represented. Each half of this caustick on each side of the axis ACK consists of two arches $KkFl$ and lki, that are convex towards one another and form a cusp at l within the circle. The arch $KkFl$ of the second caustick cuts the circle in the same point F as the first caustick does. For by the proportion above, when the points F, b coincide, or when Fc equals cb, then bd and dk are also equal. The reason of the cusp at l is this; that while bk is increasing, and then decreasing again, the point b is continually approaching towards G *. The arches KFl and lki are convex towards one another, because the emergent ray, while its point of contact k is moving from K to F, to l and to i, cuts the axis CK in greater and greater angles, till at last it emerges at i in a tangent to the circle and to the caustick too. When the focus of incident rays is nearer to the sphere than its focal distance, the secondary caustick FKk will have two asymptotes like as the primary one; and their shapes will be much alike. Fig. 465.

* Art. 483.

CHAP-

CHAPTER X.

Concerning the Rain-bow.

PROPOSITION I.

488. WHEN *a ray of light is refracted into a circle, and successively reflected within it any given number of times before it emerges out of the circle by a second refraction; let the angle of refraction be multiplyed by the number of successive reflections increased by an unite; and the excess of the resulting angle above the angle of incidence will be equal to half the angle contained under the incident and the emergent ray produced till they meet: that is, the excess abovementioned is equal to half that angle, under the incident and the emergent ray, in which the refracting circle lyes, when the number of reflections is odd; and is equal to half the other angle, under the same rays, which is the complement of the former to two right angles, when the number of reflections is even.*

Fig. 466 to 469.

*Art. 183 &c.

For let $ABCDE$ be a great circle of a sphere whose center is O, and let an incident ray SA be refracted at A to B, and be reflected from B to C; and at C let it either go out by refraction to G, or be reflected to D*; where let it either go out by refraction to H or be reflected to E; and so on. And when the number of reflections is odd, a line OR drawn through the center O and the middlemost point of reflection, will bisect the angle at R under the incident and the emergent ray produced: because the reflections and refractions on each side of the line OR are equal in number and magnitude; the chords AB, BC, CD, DE described by the reflected ray being equal to one another. And for the same reason when the number of reflections is even, a line OT, drawn through the center O perpendicular to the chord that joins the two middlemost points of reflection, will bisect one of the angles at T under the incident and the emergent ray produced; and a line TV, perpendicular to TO, will bisect the other angle under them, which is the complement of the former to two right ones. Hence the line TV is parallel to the middlemost chord, because TO is perpendicular to them both. Draw a diameter POQ parallel to the incident ray SAM, and let it cut the reflected rays BC, CD, DE produced, in β, γ, δ, respectively. Join OA, OB and in fig. 466 the sums of the three angles in each of the triangles OAB, OAR, are equal to one another; take away the common angle AOB, and the sum of the equal angles OAB, OBA in the first triangle, will be equal to the sum of the angles OAR, ORA in the second triangle. And by substracting the angle of incidence OAR or OAM from both sums, we have $2OAB-OAM = ORA = BOQ$. Hence in fig. 467. the angle STV or $P\beta C$, being an external

external angle of the triangle $OB\beta$, equals $OBC + BOQ = OAB + 2OAB - OAM = 3OAB - OAM$. Hence again in fig. 468 the angle SRO or POC, being an external angle of the triangle $OC\beta$, equals $OCB + P\beta C = OAB + 3OAB - OAM = 4OAB - OAM$. Hence again in fig. 469 the angle STV or $P\gamma D$, being an internal angle of the triangle $CO\gamma$, equals $OCD - CO\gamma = 5OAB - OAM$, throwing away two right angles. For $CO\gamma = 2$ right angles $- POC = 2$ right angles $- 4OAB + OAM$. And so forward continually. Therefore if the number of successive reflections increased by an unite be called m, it appears that $mOAB - OAM$ equals half the angle under the incident and emergent rays. *Q. E. D.*

PROPOSITION II.

489. *Things remaining as they were, let the angle of incidence increase from nothing till it becomes a right angle; and the angle under the incident and the emergent ray, after any given number of reflections called* n, *will first increase and then decrease again; and will be the greatest of all when the tangent of the angle of incidence, is to the tangent of the angle of refraction, as* n + 1 *to* 1.

For putting $m = n + 1$, we had half the angle under an incident and the emergent ray equal to the excess of $mOAB$ above OAM*; which excess, when the angles OAB, OAM are very small, will also be but small; and will increase so long as the successive increments of $mOAB$ shall exceed the contemporary increments of OAM; and will decrease again when the successive increments of $mOAB$ are exceeded by the increments of OAM; and consequently will be the greatest of all when m times the least increment of OAB is equal to once the contemporary increment of OAM; that is when the least increment of the angle of incidence OAM is to the contemporary increment of the angle of refraction OAB, and consequently the tangent of incidence is to the tangent of refraction [a], as m to 1. *Q. E. D.* Fig. 466 to 469. * Art. 448. a Art. 421.

PROPOSITION III.

490. *It is proposed to find two angles, whose sines shall be in a given ratio of* I *to* R, *and whose tangents shall be in another given ratio of* m *to* 1.

In any given line $CEDA$, let CA be to CD as I to R, and CA to CE as m to 1; with the center C and semidiameter CD describe an arch DB, cutting a circle ABE whose diameter is AE, in B; draw ABF, and joining BC, the sine of the angle CBF will be to the sine of CAF as I to R; and the tangent of CBF to the tangent of CAF as m to 1; and consequently CBF, CAF are the angles required. For in the triangle CAB the sine of the angle CBA or CBF, is to the sine of CAF, as CA to CB* Fig. 470. * Art. 221.

or

or CD, as I to R by construction. Join BE and compleat the parallelogram $CEBG$; and CG produced will cut ABF at right angles in F, because ABE is a right angle in the semicircle ABE. Therefore the lines FC, FG are tangents of the angles CBF, GBF or CAF to the radius BF;
* Euc. VI. 2. and the tangent FC is to the tangent FG as FA to FB* or as CA to CE* or as m to 1 by construction. *Q. E. D.*

491. *Corol.* 1. When parallel rays of the sun fall upon a spherical drop of rain, let the given ratio of I to R stand for the ratio of the sine of incidence to the sine of refraction; and let n be any given number of successive reflections made by every ray before it emerges out of the drop, and let $m = n + 1$; and by these propositions it appears, that half the greatest angle which any of the emergent rays can make with the incident rays, is equal to $m \times \text{ang.} CBF - CAF$. For CBF and CAF or GBF are angles whose sines are as I to R, and whose tangents are as m to 1; and consequently are the angles of incidence and refraction of that ray, whose incident and emergent parts produced contain the greatest angle.

492. *Corol.* 2. The foregoing construction for determining the angle CBF is Dr. *Halley*'s [a], and Sir *Isaac Newton*'s rule for calculating it, is this that follows. As $\sqrt{\overline{mm-1} \times RR}$ is to $\sqrt{II-RR}$, so is the tabular radius to the cosine of the angle of incidence CBF. Whence this angle and its sine are given by the tables, and from thence by the ratio of I to R the tabular sine of the angle of refraction and the angle it self are also given. The rule may thus be demonstrated. We had $CA : CB :: I : R$ and $FA : FB :: m : 1$. Hence $CA^2 = \frac{II}{RR} CB^2$, and $AF^2 = mmBF^2$; and so

* Euc. I. 47. $\frac{II}{RR} CB^2 - mmBF^2 = (CA^2 - AF^2 = FC^2 {}^* =) CB^2 - BF^2$. Hence $\frac{II}{RR} CB^2 - CB^2 = mmBF^2 - BF^2$, and $\overline{II-RR} \times CB^2 = \overline{mm-1} \times RR \times BF^2$; and by resolving this equality into a proportion, and by extracting the roots, we have $\sqrt{\overline{mm-1} \times RR} : \sqrt{II-RR} :: CB : BF ::$ radius : cosine ang. CBF.

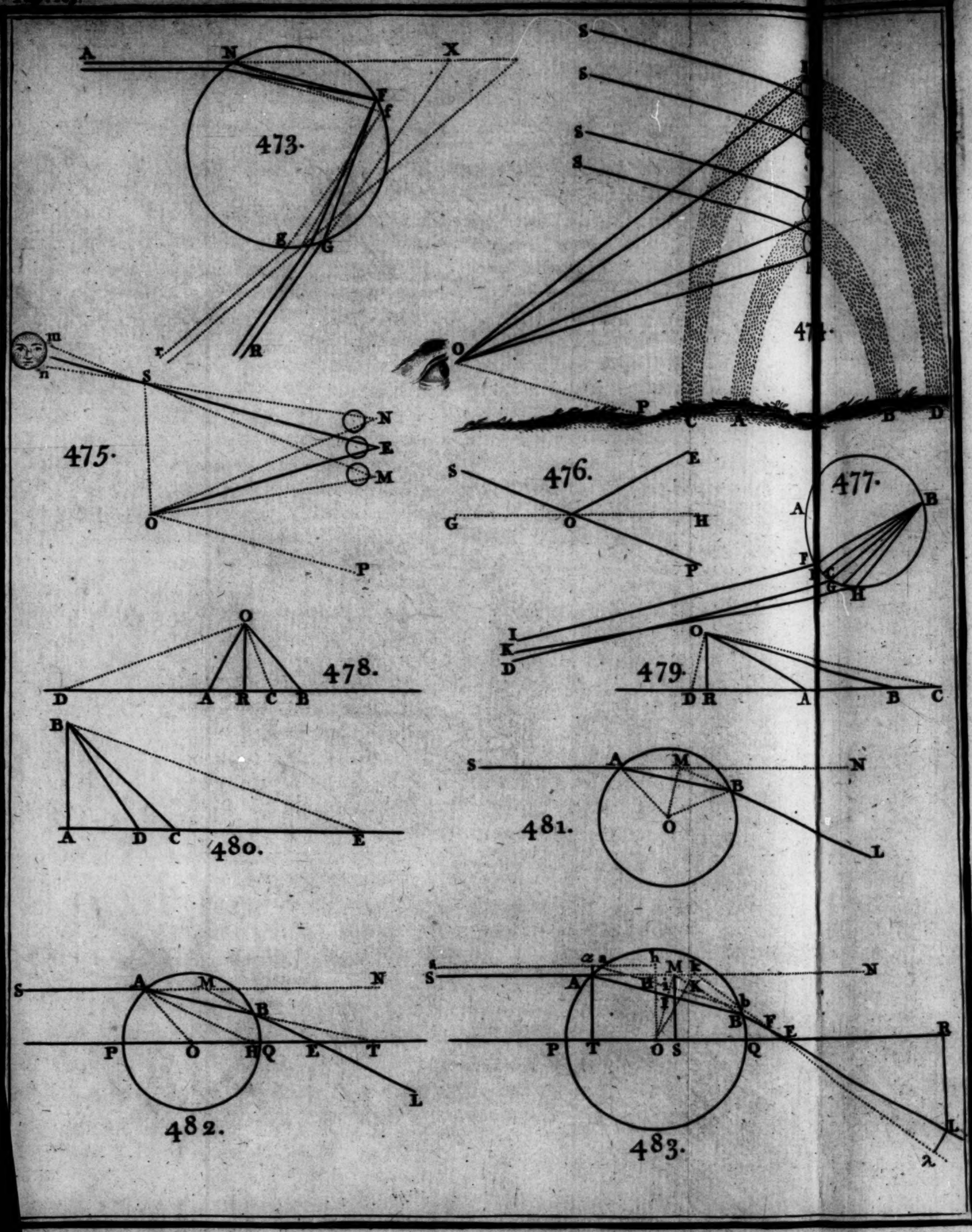
a Phil. Transf. N°. 297.

Proposition IV.

To explain the Phænomena of the Rain-bow.

Design. 493. Having premised such mathematical principles as are necessary for an exact computation of the apparent diameters and breadths of the Rain-bows, I will here subjoin Sir *Isaac Newton*'s entire explication of the colours of the bows and of the manner in which they are formed; taking the liberty here and there of making a few additions to it; for the sake of such readers as may not be so skilful as those that he generally writes to.

494.


473.
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494. This bow never appears but where it rains in the sun-shine, and may be made artificially by spouting up water which may break aloft, and scatter into drops, and fall down like rain. For the sun shining upon these drops, certainly causes the bow to appear to a spectator standing in a due position to the rain and sun. And hence it is now agreed upon that this bow is made by refraction of the sun's light in drops of falling rain. This was understood by some of the ancients, and of late more fully discovered and explained by the famous *Antonius de Dominis* Archbishop of *Spalato* in his book *de Radiis Visus & Lucis* published by his friend *Bartolus* at *Venice* in the year 1611, and written above 20 years before. For he teaches there how the interior bow is made in round drops of rain by two refractions of the sun's light, and one reflection between them; and the exterior bow by two refractions and two sorts of reflections between them in each drop of water; and proves his explications by experiments made with a phial full of water, and with globes of glass filled with water, and placed in the sun to make the colours of the two bows appear in them. The same explication *Des-Cartes* has pursued in his *Meteors* and mended that of the exterior bow. But while they understood not the true origin of colours, it is necessary to pursue it a little farther. Newt. Opt. p. 147.

495. For understanding therefore how the bow is made, let a drop of rain or any other spherical transparent body be represented by the sphere *BNFG* described with the center *C* and semidiameter *CN*. And let *AN* be one of the sun's rays incident upon it at *N*, and be thence refracted to *F*, where let it either go out of the sphere by refraction toward *V*, or be reflected to *G*; and at *G* let it either go out by refraction to *R*, or be reflected to *H*; and at *H* let it go out by refraction towards *S*, cutting the incident ray in *Y*. Produce *AN* and *RG* till they meet in *X*, and upon *AX* and *NF* let fall the perpendiculars *CD* and *CE*, and produce *CD* till it falls upon the circumference at *L*. Parallel to the incident ray *AN* draw the diameter *BQ*; and let the sine of incidence out of air into water, be to the sine of refraction as *I* to *R*. Now if you suppose the point of incidence *N* to move from the point *B* continually till it comes to *L* the arch *QF* will first increase and then decrease, and so will the angle *AXR* which the rays *AN* and *GR* contain; and the arch *QF* and angle *AXR* will be biggest when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{3RR}$, in which case *NE* will be to *ND* as 2*R* to *I*. Also the angle *AYS*, which the rays *AN* and *HS* contain will first decrease and then increase; and grow least when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{8RR}$, in which case *NE* will be to *ND* as 3*R* to *I*; and so the angle which the next emergent ray (that is the emergent ray after three reflections) contains with the incident ray *AN* will come to its limit, when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{15RR}$; in which case *NE* will be to *ND* as 4*R* to *I*. And the angle Fig. 471.

which the ray next after that emergent, (that is the ray emergent after four reflections) contains with the incident, will come to its limit, when ND is to NC as $\sqrt{II-RR}$ to $\sqrt{24RR}$; in which case NE will be to ND as $5R$ to I; and so on infinitely, the numbers 3, 8, 15, 24, &c. being gathered by continual addition of the terms of the arithmetical progression 3, 5, 7, 9, &c. The truth of all which is evident either by art. 492. or by art. 429, 430.

496. Now it is to be observed, that as when the sun comes to his tropicks, days increase and decrease but a very little for a great while together; so when by increasing the distance CD, these angles come to their limits, they vary their quantity but very little for some time together; and therefore a far greater number of rays which fall upon all the points N in the quadrant BL shall emerge in the limits of these angles than in any other inclinations. Add to this, that of all the rays which fall upon the quadrant BL, those contiguous ones can only emerge parallel to one another, which emerge in the limits of these angles; and that all other contiguous rays emerge diverging from points either behind or before the drop; and consequently will fall much thinner upon the eye, at a great distance from the drop, than the parallel rays. For those rays which converge to points behind the eye, placed at a great distance from a small drop, are not sensibly different from parallel rays. This will appear by observing that while the arch BN is continually increasing from nothing, and the angle AXR, for example, is also increasing; the successively emergent rays, being continually less and less inclined to the incident ones or to the fixt line BQ, are also successively inclined in small angles to one another; and the same thing is manifest while the angle AXR is decreasing; the successive rays being more and more inclined to PQ; consequently in the limit between the increase and decrease of this angle the contiguous incident rays must emerge parallel to one another.

Fig. 472.

497. And farther it is to be observed, that the rays which differ in refrangibility will have different limits of their angles of emergence; and by consequence according to their different degrees of refrangibility, emerge most copiously in different angles; and being separated from one another appear each in their proper colours. Add to this that although the heterogeneal rays of any slender pencil whatever, as AN, will be separated by refractions at the drop into rays $NFGR$ of one colour, and $Nfgr$ of another, as by refractions through a prism; yet these emergent rays GR, gr will not affect the eye with their distinct colours, unless they be in the limits of the angles AXR, Axr; because every where within these greatest angles, an infinite number of such coloured pencils being variously inclined to one another are mixt together, and consequently appear white or without distinct colours. And the same may be said of the rays emerging any where within the greatest angle NYS. Fig. 471.

Fig. 473.

498

498. Now what these angles are may be gathered first by computing the angles of incidence and refraction by art. 492 or 429 and 430, and then the angles *AXG*, *AYS* themselves by the 488th article. For in the least refrangible rays the sines *I* and *R* are 108 and 81 *, and thence by computation the greatest angle *AXR* will be found 42 degrees and 2 minutes; and the least angle *AYS*, 50 degrees and 57 minutes. And in the most refrangible rays the sines *I* and *R* are 109 and 81, and thence by computation the greatest angle *AXR* will be found 40 degrees and 17 minutes, and the least angle *AYS*, 54 degrees and 7 minutes. * Art. 179.

499. Suppose now that *O* is the spectator's eye, and *OP* a line drawn parallel to the sun's rays. Let *POE*, *POF*, *POG*, *POH* be angles of 40 degr. 17 min.; 42 deg. 2 min.; 50 deg. 57 min.; 54 deg. 7 min. respectively; and these angles turned about their common side *OP*, shall with their other sides *OE*, *OF* and *OG*, *OH* describe the verges of two rain-bows *AFBE* and *CHDG*. For if *E*, *F*, *G*, *H* be drops placed any where in the conical superficies described by *OE*, *OF*, *OG*, *OH* and be illuminated by the sun's rays *SE*, *SF*, *SG*, *SH*; the angle *SEO* being equal to *PEO* or 47 deg. 17. min. shall be the greatest angle in which the most refrangible rays can after one reflection be refracted to the eye; and therefore all the drops in the line *OE* shall send the most refrangible rays most copiously to the eye; and thereby strike the senses with the deepest violet colour in that region. In like manner the angle *SFO* being equal to the angle *POF* or 42 deg. 2 min. shall be the greatest in which the least refrangible rays after one reflection can emerge out of the drops; and therefore these rays shall come most copiously to the eye from the drops in the line *OF*, and strike the senses with the deepest red colour in that region. And by the same argument the rays which have intermediate degrees of refrangibility shall come most copiously from drops between *E* and *F* and strike the senses with the intermediate colours in the order which their degrees of refrangibility require; that is in the progress from *E* to *F* or from the inside of the bow to the outside in this order, violet, indigo, blue, green, yellow, orange, red. But the violet by the mixture of the white light of the clouds will appear faint and incline to purple. It may be farther observed, that all the rays but the violet in the line *SE* will emerge from *E* in a greater angle than *SEO* made by the violet, and consequently will pass below the eye; and that all the rays but the red in the line *SF* will emerge from *F* in a lesser angle than *SFO* made by the red, and consequently will pass above the eye; by which means only red will appear in the line *SF* and only violet in the line *SE*. Fig. 474.

500. Again the angle *SGO* being equal to the angle *POG* or 50 deg. 51 min. shall be the least angle in which the least refrangible rays can after two reflections emerge out of the drops; and therefore the least refrangible rays shall come most copiously to the eye from the drops in the line

line OG, and ſtrike the ſenſe with the deepeſt red in that region. And the angle SHO being equal to the angle POH or 54 deg. 7 min. ſhall be the leaſt angle in which the moſt refrangible rays after two reflections can emerge out of the drops; and therefore theſe rays ſhall come moſt copiouſly to the eye from the drops in the line OH, and ſtrike the ſenſe with the deepeſt violet in that region. And by the ſame argument the drops in the regions between G and H ſhall ſtrike the ſenſe with intermediate colours in the order which their degrees of refrangibility require; that is in the progreſs from G to H, or from the inſide of the bow to the outſide in this order; red, orange, yellow, green, blue, indigo, violet. And ſince theſe four lines OE, OF, OG, OH may be ſituated any where in the abovementioned conical ſurfaces, what is ſaid of the drops and colours in theſe lines is to be underſtood of the drops and colours every where in thoſe ſurfaces.

501. Thus ſhall there be made two bows of colours, an interior and ſtronger by one reflection in the drops, and an exterior and fainter by two; for the light becomes fainter by every reflection. And their colours ſhall lye in a contrary order to one another; the red of both bows bordering upon the ſpace GF, which is between the bows. The apparent breadth of the interior bow EOF, meaſured croſs the colours, ſhall be 1 deg. 45 min. and the breadth of the exterior, GOH, ſhall be 3 deg. 10 min. and the apparent diſtance between them, GOF, ſhall be 8 deg. 55 min. the greateſt ſemidiameter of the innermoſt, that is, the angle POF being 42 deg. 2 min. and the leaſt ſemidiameter of the outermoſt, POG, being 50 deg. 57 min.

502. Theſe are the meaſures of the bows as they would be were the ſun but a point, for by the breadth of his body the breadths of the bows will be increaſed and their diſtance decreaſed by half a degree. And ſo the breadth of the interior Iris will be 2 deg. 15 min., that of the exterior 3 deg. 40 min., their diſtance 8 deg. 25. min., the greateſt ſemidiameter of the interior bow 42 deg. 17 min., and the leaſt of the exterior 50 deg. 42 min. For let SEO be the limit of all the angles under the rays of any one colour, which coming from the center of the ſun are reflected from the drop at E to the eye at O. In the ray SE take any point S at pleaſure and let the angles ESM, ESN and alſo EOM, EON be ſeverally equal to a quarter of a degree, that is to half the apparent breadth of the ſun. And joining OS, ſince the ſums of the angles at the baſe OS, of the ſeveral triangles OSM, OSE, OSN, are equal among themſelves, their vertical angles at M, E, N are alſo equal among themſelves. Conſequently the angle SMO will be the limit of all the angles contained under the incident and emergent rays of the ſame colour as before, which came from m the higheſt point of the ſun; and SNO the limit of all the angles contained under the incident and emergent rays of the ſame colour as before, which came from n the loweſt point of the ſun. Therefore if all the rays of the ſun were of the ſame colour, or alike

Fig. 475.

alike refrangible, the apparent breadth of the bow, measured by the angle *MON*, would be but half a degree or equal to the apparent breadth of the sun measured by the angle *MSN* or *mSn*. But since his rays are differently refrangible, conceive the drop *E* to be placed any where in the inward or outward verges of the bows, above described upon supposition that the sun was but a point; and then it is manifest that the angle *EOM* must be added to the inside, and *EON* to the outside of the angles which the breadths of those bows subtend at *O*, to obtain their apparent breadths. A rain-bow is therefore a circular image of the sun reflected to the eye from the farther surfaces of innumerable drops of falling rain, and dilated in breadth by the unequal refrangibility of rays of different colours.

503. And such are the dimensions of the bows in the heavens found to be very nearly, when their colours appear strong and perfect. For once by such means as I then had I measured the greatest semidiameter of the interior iris about 42 degrees, the breadth of the red, yellow, green in that iris 63 or 64 minutes, besides the outmost faint red obscured by the brightness of the clouds, for which we may allow 3 or 4 minutes more. The breadth of the blue was about 40 minutes more besides the violet, which was so much obscured by the brightness of the clouds that I could not measure its breadth. But supposing the breadth of the blue and violet together to equal that of the red, yellow and green together; the whole breadth of this iris will be about $2\frac{1}{4}$ degrees, as above. The least distance between this iris and the exterior iris was about 8 degrees and 30 minutes. The exterior iris was broader than the interior, but so faint, especially on the blue side that I could not measure its breadth distinctly. At another time when both bows appeared more distinct I measured the breadth of the interior iris 2 deg. 10 min. and the breadth of the yellow and green in the exterior iris was to the breadth of the same colours in the interior as 3 to 2.

504. Whoever has a mind to repeat these observations after Sir *Isaac Newton* may observe, that the apparent semidiameter of the bow, (or of any ring of colours in either of the bows) is equal to the apparent altitude of its highest point added to the sun's altitude, and consequently may be measured by a common quadrant. For let *SOP* be the axis of the bows passing through the sun at *S* and the eye at *O*, *GOH* an horizontal line, *E* the highest point of any ring of either of the bows, whose apparent semidiameter *EOP* is required. It is manifest that the angle $EOP = EOH + HOP = EOH + SOG$. Fig. 476.

505. This explication of the rain-bow is yet farther confirmed by the known experiment (made by *Antonius de Dominis* and *Des-Cartes*) of hanging up any where in the sun-shine a glass-globe filled with water, and viewing it in such a posture that the rays which come from the globe to the eye may contain with the sun's rays an angle of either 42 or 50 degrees.

grees. For if the angle be about 42 or 43 degrees the ſpectator ſuppoſed at *O*, ſhall ſee a full red colour in that ſide of the globe which is oppoſed to the ſun; as it is repreſented at *F*: and if that angle be made leſs, ſuppoſe by depreſſing the globe to *E*, there will appear other colours yellow, green and blue ſucceſſively in the ſame ſide of the globe. But if the angle be made about 50 degrees, ſuppoſe by lifting up the globe to *G*, there will appear a red colour in that ſide of the globe which lyes toward the ſun: and if the angle be made greater, ſuppoſe by lifting up the globe to *H*, the red will turn ſucceſſively to other colours, yellow, green and blue. The ſame thing I have tryed by letting a globe reſt, and by raiſing or depreſſing the eye, or otherwiſe moving it to make the angle of a juſt magnitude. So far Sir *Iſaac Newton*.

506. It remains now to take notice of ſeveral ſlender rings of colours contiguous to the inſide of the firſt rain-bow. Theſe have been obſerved and particularly deſcribed by the Reverend Dr. *Langwith* in the Philoſophical Tranſactions N°. 375. His beſt obſervation was this. The colours of the primary rain-bow were as uſual, only the purple very much inclining to red and well defined. Under thiswas an arch of green, the upper part of which inclined to a bright yellow, the lower to a more duſky green: under this were alternately two arches of rediſh purple and two of green: under all a faint appearance of another arch of purple which vaniſhed and returned ſeveral times ſo quick that we could not fix our eyes upon it. Thus the orders of the colours were.

I. Red, orange, yellow, green, light blue, deep blue, purple.
II. Light green, dark green, purple.
III. Green, purple.
IV. Green, faint vaniſhing purple.

Thus we had four orders of colours and perhaps the beginning of a fifth; for I make no queſtion but that which I call purple (when very red) is a mixture of purple of the upper ſeries with the red of the next below it: and the green a mixture of the intermediate colours. There are two things, which will deſerve to be taken notice of, as they may perhaps direct us to the ſolution of this curious phænomenon. Firſt that the breadth of the firſt ſeries ſo far exceeded that of all the reſt, that as near as I could judge, it was equal to them all taken together. Secondly that I have never obſerved theſe inner orders of colours in the lower parts of the rain-bow, though they have been incomparably more vivid than the upper parts, under which theſe colours have appeared. I have taken notice of this ſo very often that I can hardly look upon it to be accidental; and if it ſhould prove true in general, it will bring the diſquiſition into a narrow compaſs. For it will ſhew that this effect depends upon ſome property, which the drops retain while they are in the upper part of the air, but loſe as they

they come lower, and are more mixed with one another. I am of opinion that the rain-bow ſeldom appears very lively, without ſome of thoſe orders of colours; and that the ſuppoſed exact agreement between the colours of the rain-bow and thoſe of the priſm, is the reaſon that it has been ſo little obſerved. So far Dr. *Langwith*. To which Dr. *Pemberton* ſubjoins the following theory, printed in the Tranſactions N°. 375. and in his view of Sir *Iſaac Newton*'s Philoſophy p. 401.

507. Sir *Iſaac Newton* has obſerved that in glaſs which is poliſhed and quick-ſilvered, there is an irregular reflection made, whereby ſome ſmall quantity of light is ſcattered from the principal reflected beam. If we allow the ſame thing to happen in the reflection whereby the rain-bow is cauſed, it ſeems ſufficient to produce the appearance abovementioned. Let *AB* repreſent a globule of water, *B* the point from which the rays of any determinate ſpecies, being reflected to *C* and afterwards emerging in the line *CD*, would proceed to the eye and cauſe the appearance of the colour in the rain-bow, which appertains to this ſpecies. Here ſuppoſe that beſides what is reflected regularly, ſome ſmall part of the light is irregularly ſcattered every way, ſo that from the point *B* beſides the rays that are regularly reflected from *B* to *C*, ſome ſcattered rays will return in other lines, as in *BE*, *BF*, *BG*, *BH*, on each ſide the line *BC*. Farther it has been obſerved by Sir *Iſaac Newton* that the rays of light in their paſſage from one ſuperficies of a refracting body to the other, undergo alternate fits of eaſy tranſmiſſion and reflection, ſucceeding each other at equal intervals: inſomuch that if they reach the farther ſurface in one ſort of thoſe fits, they ſhall be tranſmitted; if in the other kind of them they ſhall rather be reflected back. Whence the rays that proceed from *B* to *C*, and emerge in the line *CD*, being in a fit of eaſy tranſmiſſion, the ſcattered rays that fall at a ſmall diſtance without theſe on either ſide (ſuppoſe the rays that paſs along the lines *BE*, *BG*) ſhall fall on the ſurface in a fit of eaſy reflection and ſhall not emerge; but the ſcattered rays, which paſs at ſome diſtance without theſe laſt, ſhall arrive at the ſurface of the globule in a fit of eaſy tranſmiſſion and break through that ſurface. Suppoſe theſe rays to paſs in the lines *BF*, *BH*, the former of which ſhall have had one fit more of eaſy tranſmiſſion and the latter one fit leſs, than the rays that paſs from *B* to *C*. Now both theſe rays when they go out of the globule, will proceed by the refraction of the water in the lines *FI*, *HK*, that will be inclined almoſt equally to the rays incident upon the globule, which come from the ſun: but the angles of their inclination will be leſs than the angle in which the rays emerging in the line *CD* are inclined to thoſe incident rays. And after the ſame manner rays ſcattered from the point *B* at a certain diſtance without theſe will emerge out of the globule, while the intermediate rays are intercepted; and theſe emergent rays will be inclined to the rays incident upon the globule in angles ſtill

Opt. Book. II. Part 4. at the beginning.

Fig. 477.

Opt. Book II. Part. 3. prop. 12.

less than the angles, in which the rays FI and HK are inclined to them; and without these rays will emerge other rays that shall be inclined to the incident rays in angles still less. Now by this means may be formed of every kind of rays, besides the principal arch, which goes to the formation of the rain-bow, other arches within every one of the principal of the same colour, though much more faint; and this for diverse successions, as long as these weak lights, which in every arch grow more and more obscure, shall continue visible. Now as the arches produced by each colour will be variously mixed together, the diversity of colours observed in these secondary arches may very possibly arise from them. In the darker colours these arches may reach below the bow and be seen distinct: in the brighter colours these arches are lost in the inferior part of the principal light of the rain-bow; but in all probability they contribute to the red tincture which the purple of the rain-bow usually has, and is most remarkable when these secondary colours appear strongest. However these secondary arches in the brightest colours, may possibly extend with a very faint light below the bow and tinge the purple of the secondary arches with a redish hue. The precise distances between the principal arch and these fainter arches depend on the magnitude of the drops, wherein they are formed. To make them any degree separate it is necessary the drop be exceeding small. It is most likely they are formed in the vapour of the cloud, which the air, being put into motion by the fall of the rain, may carry down along with the larger drops: and this may be the reason why these colours appear under the upper part of the bow only, this vapour not descending very low. As a farther confirmation of this, these colours are seen strongest when the rain falls from very black clouds, which cause the fiercest rains by the fall whereof the air will be most agitated. So far Dr. *Pemberton.*

To the like alternate fits of easy transmission and reflection in the passage of light through the least globules of water, Sir *Isaac Newton* ascribes those little rings of colours which sometimes appear round about the sun and moon. Opt. p. 288. obs. 13.

LEMMA

508. The tangent of the sum of two angles, is to the sum of their tangents, as the square of the radius, to the square of the radius diminished by the rectangle under the tangents: and the tangent of the difference of two angles, is to the difference of their tangents, as the square of the radius, to the square of the radius increased by the rectangle under the tangents.

Fig. 478, 479. Let RA and RB be tangents of two angles ROA, ROB. Then as AB, the sum or difference of the tangents, is to AO, the secant of either of the angles, so let AO be to AC, to be placed from A towards B. Again as RC is

is to RO, so let RO be to RD; and RD will be the tangent of the sum or difference of the two angles ROA, ROB. For joining CO, by the first of these proportions the triangles AOB, ACO will be equiangular[a]; and so the angle AOB is equal to ACO, or to ROD by the second proportion[b]. Hence in fig. 478, because $AC = \frac{AOq}{AB} = \frac{RAq + ROq}{RB + RA}$, we have $RC = (AC - AR =) \frac{RAq + ROq}{RB + RA} - RA = \frac{ROq - RB \times RA}{RB + RA}$; whence $RD = \frac{RB + RA}{ROq - RB \times RA} \times ROq$. By a like process fig. 479. we have $AC = \frac{RAq + ROq}{RB - RA}$; whence $RD = \frac{RB - RA}{ROq + RB \times RA} \times ROq$. Q. E. D.

a Euc. VI. 6.

b Euc. VI. 8.

509. *Corol.* 1. Hence the tangent of the sum of any number of given angles, or the tangent of any multiple of a given angle, may be easily computed. Put $RO = r$, $RA = a$, $RB = b$, then the tangent of the sum of the angles whose tangents are a and b, that is $RD = \frac{b + a}{rr - ab} \times rr$; call this tangent x; then for the same reason, the tangent of the sum of this last angle and of a third angle, whose tangent is c, is $\frac{x + c}{rr - xc} \times rr$ or (by substituting the value of x) $\frac{rr \times \overline{a + b + c} - abc}{rr - ab - ac - bc}$, the tangent of the sum of three angles whose tangents are a, b, c; and so on.

510. *Corol.* 2. Now put $a = b = c$; and for the tangent of a double angle we have $\frac{2a}{rr - aa} rr$; and for the tangent of a treble angle $\frac{3arr - a^3}{rr - 3aa}$; and so on.

PROPOSITION V.

511. *The apparent semidiameter of any rain-bow, or the greatest angle under an incident and the emergent ray after any given number of successive reflections, being given; to find the ratio of refraction.*

Let m be the given number of successive reflections increased by an unite, and supposing the angles ABC, ABD to be the angles of incidence and refraction sought, let the angle $ABE = m \times ABD$, and the angle CBE, or $m \times ABD - ABC$, will be half the given angle under the incident and the emergent ray after $m - 1$ reflections[c]. Put the common radius $AB = r$, the unknown tangent of refraction $AD = a$, and the tangent of incidence $AC = ma$*, also $AE = x$, and t for the tangent of the given angle CBE answering to the radius r. Then by the lemma $t : x - ma :: rr : rr + xma$; whence $t = \frac{x - ma}{rr + xma} rr$.

Fig. 480.

c Art. 488.

* Art. 491.

Case 1. In the first rain-bow $m=2$; whence $t=\frac{x-2a}{rr+2xa}rr$, and by art. 520, $x=\frac{2a}{rr-aa}rr$ the tangent of $2ABD$. Substitute this value for x in the former equation and by reduction it becomes $a^3-\frac{1}{2}taa-\frac{1}{2}trr=0$. By resolving this equation the tangent a of the angle of refraction will be given, and the tangent of the angle of incidence $AC=2a$ by art. 491. whence the ratio of their sines is given by the tables.

Case 2. In the second rain-bow $m=3$, whence $t=\frac{x-3a}{rr+3xa}rr$, and by art. 510, $x=\frac{3arr-a^3}{rr-3aa}$, the tangent of $3ABD$. Substitute this value for x and you will find $a^4+\frac{8}{3}\frac{rr}{t}a^3-2rraa*-\frac{1}{3}r^4=0$; or putting $T=\frac{rr}{t}$ the tangent of half the angle of this bow[a], $a^4+\frac{8}{3}Ta^3-2rraa*-\frac{1}{3}r^4=0$. The same method serves for other bows to infinity.

a Art. 488.

512. *Corol.* In the first case putting T for $2a$ or AC the tangent of the angle of incidence, and substituting $\frac{1}{2}T$ for a in the former equation $a^3-\frac{1}{2}taa-\frac{1}{2}trr=0$, it is changed to this $T^3-3tTT-4rrt=0$, the same as Dr. *Halley*'s, who proposed this problem as an expeditious method for finding the ratio of refraction in any fluid, by observing (when the sun is low and shines very bright) the angle under an incident and the emergent ray from a drop of any fluid hanging at the end of a capillary tube. See his examples Phil. Trans. N°. 267. and also the Rev^d^. Dr. *Morgan*'s Dissertation upon the Rain-bow among the Notes upon *Rohault*'s Physicks. P. 3. Ch. 17.

Having done with the rain-bows I proceed now to consider the rays which emerge from a drop after two refractions without any reflection, in order to account for halo's.

Proposition VI.

Fig. 481.

513. *When a ray of light* SABL *is refracted through a sphere at* A *and* B, *without any intermediate reflection, the angle* LMN, *under the incident ray* SAMN *and the emergent ray* LBM *produced, is equal to twice the excess of the angle of incidence* OAM *above the angle of refraction* OAB: *and by consequence increases perpetually while the angle of incidence increases.*

For the external angle LMN is equal to the sum of the internal angles MAB, MBA; which are equal to each other, because of the equal refractions at A and B; as appears by conceiving a ray to go out both ways along the chord AB. And one of these angles equals the excess of OAM above OAB: and since the increments of OAM are always bigger than those of OAB *, the excesses of the larger increments above the smaller will continually augment the angle BAM, and consequently the whole angle LMN. *Q. E. D.*

* Art. 421.

514.

514. *Corol.* 1. Draw a diameter *POQE* parallel to the incident ray *SAM*; and let it cut the emergent ray *BL* in *E*; and while the angle *LMN* increaſes from nothing, the line *ME* will decreaſe perpetually. For let the parallelogram *EMAH* be compleated, and let *AB* and *PQ* be produced till they meet in *T*; and the equiangular triangles *AMB*, *BET*, *AHT* will be equicrural. Now if *ME* or *AH* be ſuppoſed to increaſe or only to keep the ſame while the oppoſite angle *AOH* decreaſes, the adjoining ſide *HO* muſt increaſe, and conſequently the ſum of *AH* and *HO* or of *HT* and *HO* that is *OT* muſt alſo increaſe. But by art. 476. the line *OT* decreaſes perpetually; therefore the ſuppoſition we made that *AH* or *ME* increaſed or kept the ſame is falſe; and by conſequence it decreaſes perpetually while the angle of incidence increaſes; or while the angle *LMN* increaſes [a]. Fig. 482. a Art. 513.

515. *Corol.* 2. Hence while the angle *LMN* increaſes, the line *OE* decreaſes perpetually. For it is eaſy to ſhew that *ME* and *OE* are equal.

PROPOSITION VII.

516. *When parallel rays fall upon the ſurface of a ſphere and emerge from it after two refractions without an intermediate reflection, their denſity at the eye of a ſpectator placed at a great diſtance from the ſphere, will decreaſe perpetually, while the angles under the incident and the emergent rays increaſe.*

For ſuppoſing the ſame lines as before, let *SabFλ* be the neareſt ray to *SABFL*, and let them croſs each other in *F*, and then fall perpendicular upon *Lλ* ſuppoſed to be the diameter of the pupil of the eye. From the center *O* draw *OHb* perpendicular to the incident rays produced, *OIi* perpendicular to the refracted rays *AB*, *ab*; and *OKk* perpendicular to the emergent rays; and alſo *LR*, *MS*, *AT* perpendicular to the diameter *PQ*, drawn parallel to the incident rays; and let *TA* produced cut *Sa* in *α*. Then ſince *OI* is to *OH* and to *OK* as the leſſer of the ſines of incidence and refraction to the bigger; and alſo *Oi* to *Ob* and to *Ok* in the ſame ratio; disjointly we have *Ii* to *Hb* and to *Kk* in the ſame given ratio; and conſequently *Hb* equal to *Kk*. But the triangles *KFk*, *LFλ* are equiangular, and alſo *MES* and *LER*. Therefore $A\alpha$ or Hb or $Kk : L\lambda :: FK : FL$, and AT or $MS : LR :: EM : EL$; and by conſequence $A\alpha \times AT : L\lambda \times LR :: FK \times EM : FL \times EL$. Now ſuppoſing the whole figure to be turned about the axis *PQ*; ſince the ſame rays that paſs through the ring deſcribed by *Aα* will alſo paſs through the ring deſcribed by *Lλ*; ſuppoſing none of them to be ſtopt in the ſphere, it follows that their denſity at *L* is to their denſity at *A* reciprocally as the rings; that is directly as $A\alpha \times AT$ to $L\lambda \times LR$, or as $FK \times EM$ to $FL \times EL$ (becauſe the peripheries deſcribed by *A* and *L* are as their ſemidiameters *AT*, *LR*.) And the denſity of the incident rays being every where the ſame, the denſity Fig. 483.

of the emergent rays at L is as $FK \times EM$ directly and as $FL \times EL$ inversely; and by consequence directly as $FK \times EM$, when the eye at L is remote; because the points E, F never go far from the sphere[a]. Now while the angle LMN increases, the line FK decreases perpetually[b], and so does EM[c], and therefore the density of the rays falling upon a remote eye at L, perpetually decreases. *Q. E. D.*

a Art. 432. 515.
b Art. 420. 432.
c Art. 514.

This proposition may also be demonstrated in distinct parts as follows.

517. 1st. The density of the rays which fall perpendicular upon the line $L\lambda$ in any one plane of emergence, is as FK directly and FL inversely; and when L is remote, directly as FK. For supposing all the incident rays to be transmitted from $A\alpha$ to $L\lambda$, their density in $L\lambda$ will be to their density in $A\alpha$ as ($A\alpha$ or Hh or Kk to $L\lambda$, or as) FK to FL: and the line FL will be invariable when L is remote, because the focus F never goes far from the sphere[d].

d Art. 432.

518. 2ly. The density of the rays which fall upon the periphery described by the point L, turned about the axis PQ, is as EM directly and EL inversely: and when L is remote, directly as EM. For supposing all the incident rays to be transmitted, their density in the periphery described by the point L, will be to their density in the periphery described by the point A, reciprocally as these peripheries, that is directly as the semidiameter AT or MS, to the semidiameter LR, that is as EM to EL: and the line EL will be invariable when L is remote, because the point E never goes far from the sphere[e].

e Art. 515.

519. 3ly. It is observable that as F is the focus of those rays which emerge from the sphere in any one plane of incidence and emergence, so E is the focus of those that emerge in a conical surface described by the revolution of EL about the axis OE, to which surface all the planes of emergence are perpendicular.

520. 4ly. When a solid pencil of rays falls upon a plane surface and are uniformly dense in every part of it; we may consider the incident rays as consisting of innumerable physical planes of rays, which fall upon the surface in as many physical lines, parallel to one another. And then if their density in any one of these lines be given, their density upon the surface will be as the density of the lines; and if the density of the lines be given, the density of the rays upon the surface will be as their density in any one line upon it; and consequently if neither the one nor the other be given, the density of the rays upon the surface will be as their density in any one line and as the density of the lines conjointly.

521. 5ly. To apply this rule to the density of the rays upon the circular ring described by the revolution of $L\lambda$; since its breadth $L\lambda$ is supposed very small in comparison to its diameter, the planes of emergence will divide any small part of it into lines, as $L\lambda$, nearly parallel to one another. And by consequence the density of the rays in this little part, will be as

their

their density in any one line $L\lambda$ and as the density of the lines $L\lambda$ conjointly; that is as $\frac{FK}{FL} \times \frac{EM}{EL}$ by the 1st and 2d of these rules; and when L is remote, as the rectangle $FK \times EM$, as before.

522. 6ly. Now let heterogeneal rays be refracted through the sphere; and since the whole quantity of refraction increases gradually as the rays go farther and farther from its center[a], these heterogeneal rays will therefore be gradually more and more separated from one another[b]; and upon this account will fall thinner upon the eye at $L\lambda$ than if they were all homogeneal.

a Art. 513.
b Art. 171.

523. 7ly. We have all along supposed that all the incident rays are transmitted through the sphere. But it is certain that some of them are reflected both at the first and second surface, as in the case of the rain-bows. And it is a common observation, that as rays fall more obliquely upon any surface, the more of them will be reflected; and therefore the rays that go farther and farther from the center of the drop, and consequently fall more and more obliquely upon its surfaces, will be more and more diminished by reflections; and so the emergent light will decrease still faster, or in a greater proportion than of the decrease of the rectangle $FK \times EM$.

524. 8ly. If the sun be viewed through so large a sphere as to subtend an angle at the eye as large or larger than the sun it self would subtend at the naked eye; his body will appear biggest through the sphere when it is held in a line drawn directly from the eye to the sun; and will appear to decrease gradually while the sphere is moved sideways from that line. This shall be demonstrated in another place, and may be easily tryed by looking at a lighted candle or any bright object through a glass full of water. In this case the apparent brightness of the sun, or the real brightness of his picture upon the retina, would vary directly as the density of the rays in any one pencil when they fall on the eye, and inversely as the magnitude of the picture: because if the magnitude of the picture was invariable, its brightness would be as the density of the rays in any one pencil; and if the density was invariable, its brightness would be inversely as its magnitude. But since the angle subtended at the eye by a globule in a cloud has no sensible magnitude, the picture of the sun upon the retina, formed by rays that come through it, can have no variety of magnitudes; being always but a point which affects the sense not by its magnitude but by its brightness only. Therefore while this globule is moved sideways the brightness of the sun, seen through it, varies directly as the density of the rays at the eye which came from a single point of the sun; because the density of the rays that come at once from all the points of the sun varies in the same proportion as their density in any one pencil varies.

525. 9ly. Therefore when the sun shines upon a large cloud of such globules, the light will appear strongest in those globules which lye directly

between

between the eye and the sun, and will decay more and more in the globules that lye farther and farther from that line or from the apparent place of the sun. In this conclusion I agree with *Hugenius* in his dissertation concerning Corona's and Parhelia, printed in the next chapter; though I differ from him in the reason he gives for it, namely because the drops that lye nearest to the sun make the largest images of him[a]; (maximam solis imaginem exhibent); whereas these images being less than the globules have no sensible apparent magnitude; and therefore appear brighter or duller only because the rays fall thicker or thinner upon the eye.

a Art. 530.

526. Neverthelesss to think different from so great a master of reason as Sir *Isaac Newton*, is but little better than a degree of uncertainty; or at least a sort of pain which the mind would willingly be freed from. I have therefore been very particular in demonstrating this conclusion because I cannot reconcile it with one of his[b]. Where he says that the light which comes through drops of rain by two refractions without any reflection ought to appear strongest at the distance of about 26 degrees from the sun; [that is when the angle *LMN* is about 26 degrees;] and to decay gradually both ways as the distance from him increases and decreases. And that the same is to be understood of light transmitted through spherical hail-stones. And if the hail be a little flatted, as it often is, the light transmitted may grow so strong at a less distance than that of 26 degrees as to form a halo about the sun or moon. I have already observed that *Hugenius* was of a contrary opinion; and besides he tells us in another place of his dissertation that though he had examined all the reflections and refractions of the sun's rays in globules of water, yet he could find no cause at all for the appearance of a ring of light about him of the usual bigness of a halo[c]. And thence he began to suspect that they are formed by hail-stones that have opake globules of snow in their centers. In this opinion Sir *Isaac Newton* seems to agree with him. For he goes on in these words[d], —which halo, as often as the hail-stones are duly figured, may be coloured; and then it must be red within by the least refrangible rays, and blue without by the most refrangible ones; especially if the hail-stones have opake globules of snow in their centers to intercept the light within the halo (as *Hugenius* has observed) and make the inside thereof more distinctly defined than it would otherwise be. For such hail-stones, though spherical, by terminating the light by the snow, may make a halo red within and colourless without, and darker within the red than without as halo's use to be. For of those rays which pass close by the snow, the rubiform will be least refracted, and so come to the eye in the directest lines. The light which passes through a drop of water after two refractions and three or more reflections, is scarce strong enough to cause a sensible bow [as appears by the different strength of the first and second rain-bows] but in those cylinders of ice by which *Hugenius* explains the parhelia it may perhaps be sensible. This is

b Opt. p. 155.

Fig. 481.

c Art. 528.

d Opt. p. 155.

is all I can find of Sir *Isaac Newton*'s concerning this subject, excepting another short passage where he seems to suggest a reason why halo's appear oval, an instance of which he had mentioned just before; and then adds these words. By its being oval, and remoter from the moon below than above, I conclude that it was made by refraction in some sort of hail or snow floating in an horizontal posture, the refracting angle being about 58 or 60 degrees [a]. In this also I must confess I cannot rest satisfied; because all halo's, though formed by spherical hail with spherical globules of snow in their centers, must appear oval in the position he mentions upon account of the flattish figure of the sky, as it is explained in the 167th article.

[a] Opt. p. 291.

CHAPTER XI.

Concerning Corona's and Parhelia, commonly called Halo's and Mock-Suns.

Design.

IN this chapter I have translated *Hugenius*'s whole dissertation upon this curious subject; and as this excellent Author intended to demonstrate the more difficult things at the end of all, which the learned Editors of his Posthumous Works have supplyed in few words, I have done the same more at large in an Appendix to this chapter; which contains also the observations referred to in this dissertation.

I. Corona's what and how large.

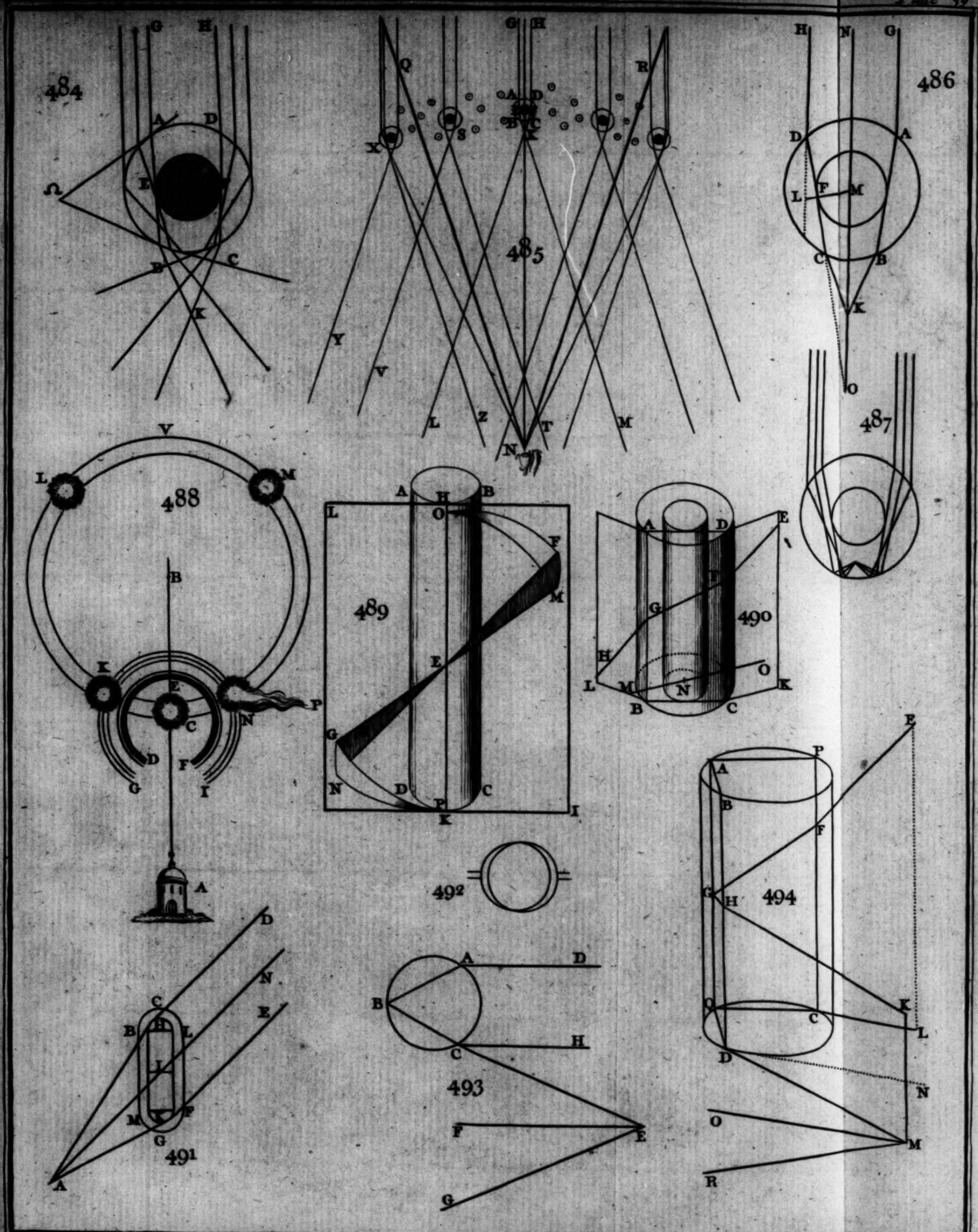
527. Though the causes of corona's and parhelia, which shall here be assigned are much alike, yet I will first treat of corona's, because they are easier to be explained; and being understood, will facilitate the explanation of parhelia. Now corona's are circular rings of light, which by day appear round about the sun, and sometimes by night about the moon; sometimes white, and sometimes, when brightest, adorned with various colours like the rain-bow. Their apparent diameter is generally about 44 or 45 degrees, but sometimes a larger sort have appeared, about 90 degrees in diameter or more. It is said that a great many such rings have appeared together, with the sun in their centers. I my self have often observed those rings of 45 degrees; the first of which, being the most lively, gave me occasion to take notice of the most remarkable appearances in it, in order to attempt their explication. I observed this ring was made up of various colours, but fainter than those in the rain-bow; that its inward edge was red and its outward a pale blue, much inclining to white; that the space within the ring (called the area) was darker than that on the outside of it, which appeared every way pretty clear, excepting that it was covered with some small pellucid and whitish clouds. I found the diameter of the ring about 45 degrees, by a gross manner of measuring it, which was by holding my cane at arms length and observing what part

of

of it covered the diameter of the area; and thence I computed the angle as near as I could. This obſervation was made 30. Mar. 1652. I remember after this obſervation I looked into what *Des Cartes* had wrote concerning the cauſes of corona's. He would have them to be generated by the rays of the ſun refracted through flat ſtars of pellucid ice: which I did not like, becauſe it follows from thence, as he himſelf confeſſes, that the ſpace within the crown ſhould appear brighter than that without, contrary to what I had obſerved.

II. Whence cauſed.

528. I therefore thought it proper to ſearch after ſome other cauſe of this appearance, though *Des Cartes* aſſerts, *that no other can be found in the heavens to produce any thing like it*. I examined all the refractions and reflections of the ſun's rays through globules of water, but found nothing that could cauſe a circle of that bigneſs. I therefore framed to my ſelf other figures conſiſting of congealed drops, but in vain; ſo hard it is to form an idea of a thing never ſeen. But being drawn into the ſame thoughts by the appearance of 5 ſuns at *Warſaw* in the year 1658; and being diligently attentive to the figure of the corona deſcribed above, at laſt I hit upon their true cauſe; and a little after upon that of the parhelia. For I make no ſcruple to call thoſe cauſes true, whoſe effects agree ſo exactly with obſervations, as to make it ſeem unneceſſary to ſearch for others. Conſidering therefore that certain particles flying about among the clouds afforded materials for theſe meteors; as appeared ſufficiently from hence, that the corona kept in the ſame place though the clouds were diſperſed; conſidering alſo that the dulneſs of the ſky within the corona was an argument that the particles within that circle did not tranſmit the ſun's light ſo copiouſly as when they flew out of it; I ſoon apprehended that the corona might be formed, if each of theſe particles were round globules conſiſting of a coat or ſhell of tranſparent ice or water, that contained an opake kernel within it. For I eaſily perceived, if a great number of ſuch particles were diſperſed between us and the ſun, that thoſe which lay within a certain diſtance from the ſun, could tranſmit no rays to the eye, whereas others more remote would let them paſs; as will appear preſently. Now it is not only probable but matter of fact, that ſuch ſort of hail-ſtones are diſperſed among the clouds; becauſe as *Des Cartes* atteſts, they ſometimes fall down upon the ground. For ſpeaking in his Meteors about the formation of ſuch particles, he ſays, *hence it comes to paſs, that as the outward ſhell of ſuch particles uſually conſiſt of a continued pellucid ice, there is often found ſome ſnow at their centers, as appears by breaking them*. Nor will it appear improbable that particles of ſnow ſhould ſtick in the middle of globules of water, conſidering that theſe globules are ſuſtained by exhalations or vapours raiſed upwards; by which means the water is hindered from deſcending and from deſerting the ſnow in its center. Neither is there any reaſon, why the kernels which are of the ſame gravity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gravity as the water, should recede from it either by ascending or descending; at least those hail-stones which descend to the earth do plainly prove the matter to be fact. Now allowing them to be sustained by ascending exhalations, it seems necessary that they should be very small, perhaps not so big as grains of turnepseed; nevertheless this smalness does not hinder the perfection of their figures, nor the just proportion between the bulk of the whole and of the kernel within, which I shall inquire into by and by. Having made it appear that such mixed corpuscles are formed in the air, I shall shew afterwards how all the appearances of corona's are produced by them.

529. The better to understand this matter, let us consider one of these globules under a larger form, in order to apprehend how the rays of the sun are refracted through it. Let *ABCD* represent a globule, with a kernel of snow *EF* in the middle of it, and let us suppose the rays coming from *GH* to fall upon the side *AD*. It is manifest they will so be refracted at the surface *AD* as to bend inwards; whence it follows that a great number of them must strike upon the kernel *EF*. Let *GA* and *HD* be the rays which after refraction touch the sides of the kernel *EF*, and let them be refracted again at *B* and *C*, and emerge in the lines *BK*, *CK* crossing each other in the point *K*; whose nearest distance from the globule is somewhat less than its semidiameter [a]. Therefore if *BK* and *CK* be produced towards *M* and *L*, it follows that no light coming from the sun through the globule, can proceed to the eye any where placed within the angle *LKM*, or rather within the cone represented by *LKM*. Because all the rays which pass by the kernel, will cross each other under a greater angle than *LKM*, and at a point nearer to the globule than the point *K* *, and therefore will pass by the eye placed any where within the angle *LKM*, but will fall upon it when placed out of that angle. What is said of this globule, belongs to all others of the same shape. That is to say every one of them carries a shady cone behind it, in which the eye being placed can perceive no light coming through the globule, but being removed out of it, will immediately perceive the light of the sun. Let the eye be at *N*, and conceive a cone whose vertex is *N* and whose sides *NR*, *NQ*, are parallel to the sides *KL*, *KM* of the former cone. It is certain that none of the globules which include equal kernels of snow to that in the globule *ABCD*, and which are placed any where within the cone *QNR*, can direct the rays of the sun towards the eye at *N*. For if from any such globule as *S*, two lines *SV* and *ST* be drawn parallel to *KL* and *KM*, representing in this figure the sides of a shady cone behind *S*, it appears that the eye at *N*, in the vertex of the cone *QNR*, falls within the cone *VST*, and consequently receives no rays coming through the globule *S*. The same may be said of all the globules within the cone *QNR*, because

III. In what manner. Fig. 484.

a Art. 61, 62.

Fig. 485.

* Art. 512. 513.

Fig. 485.

the eye at *N* is placed within the ſhady cone belonging to every one of them, as appears by the figure.

IV. The dark area with light about it, proved.

530. But if any other globule, as *X*, lying out of the cone *QNR*, be taken into conſideration, it appears that the eye at *N* lyes out of its ſhady cone *YXZ*; and conſequently thoſe rays of the ſun, which are more refracted through the globule *X* than the ray *XZ*, will come to the eye; ſo that this globule and all others out of the cone *QNR* will appear enlightened; whereas thoſe within it will appear obſcure. It is plain therefore that a certain area or ſpace quite round the ſun ought to appear dark or obſcure; and that the ſpace round about this area muſt appear more luminous; and more ſo in the parts that are neareſt to the obſcure area. Becauſe it might eaſily be demonſtrated that the globules that are neareſt to the cone *QNR* exhibit the largeſt image of the ſun[a]. It is plain alſo that a corona ought to be produced in the ſame manner, whatever be the ſun's altitude, by reaſon of the ſpherical figure both of the globules and of the kernels within them. Whoever has a mind to make a proper experiment to repreſent this appearance, let him expoſe to the ſun a thin glaſs bubble filled with water, that has an opake ſphere in the middle of it, which may eaſily be contrived; and he will find, that he cannot ſee the ſun's image in it, unleſs he removes it to a certain diſtance from the line drawn from his eye to the ſun: and that as ſoon as he perceives the light, the image of the ſun will immediately appear the brighteſt, and coloured red, for the ſame reaſon as in a glaſs priſm or rather a water priſm, if ſuch a one could be made. For the ray *GA*, for example, ſuffers the ſame refractions, at its ingreſs into the globule at *A* and at its egreſs at *B*, as it would do in paſſing through a priſm, whoſe ſides *A*Ω, *B*Ω touch the globule at *A* and *B*. And the kernel *EF*, which terminates the light on one ſide, conduces much to the appearance of the colours of the rays *AB* which paſs by it. But I have no deſign to inquire into the cauſes why theſe colours appear through priſms. I confeſs I am entirely ignorant of their production, nor do I believe that any one can diſcover it till we receive ſome greater light into the knowledge of nature. All that I would obſerve is that the red colour which appears in the glaſs bubble ought alſo to be ſeen in the globules that are neareſt to the outſide of the cone *QNR*; which growing bigger in the remoter globules produces other colours as in the rain-bow. I judge the breadth of the corona, equal to the apparent ſpace through which the colours are ſpread when the drops are moſt enlightened; although in reality the breadth is only terminated by the ſnow on the inſide next to the ſun and not on the outſide; juſt as in the rain-bow, which alſo, as *Des Cartes* has explained it, is only terminated on the red ſide and not on the other. Theſe corona's do alſo often appear about the moon for the ſame reaſons as about the ſun: but the colours are generally ſo weak as to appear only white: theſe white corona's I have often ſeen about the ſun,

a See Art. 525.

Fig. 484.

sun, when the space within them appeared scarce darker than that without. Now this happens when there is less plenty of such globules. For the more there are the more lively the colours appear, as is manifest from observations; by which it appears, that when the area within the corona is darker, that is when the globules are more numerous, the colours appear more vivid. But the colours within the corona appear more lively when parhelia and paraselenæ, that is, mock-suns and mock-moons appear with them, for another reason to be given when I treat of these appearances.

531. Let us now consider the apparent diameter of the corona, which, as we said before, is generally about 45 degrees. This depends upon the magnitude of the dark kernel *EF*. For the greater it is in comparison to the whole globule *ABCD*, so much the greater is the angle *BKC*, to which the angle of the cone *QNR* is equal and measures the apparent diameter of the corona, as we have shewn above. Besides the rays *GABK*, *HDCK*, passing through the globule *ABCD*, and touching the kernel in *E* and *F*, draw from the center *M* the line *MKO*, passing through *K* the intersection of the rays, and meeting the line *DC*, produced in *O*: let the lines *HD* and *KC* produced meet in *L* within the globule. Now in the triangle *DLC*, the angles *FDL*, *FCL* are equal, because the ray *HD* is refracted at *D*, in entering into the globule, just as much as at *C*, in going out of it. But the angle *FDL* is equal to *DOK*, because *HL* and *DO* are parallel; therefore in the triangle *KOC*, the angles *KOC*, *KCO* are also equal; and both together are equal to the angle *CKM*, which is half of *CKB*. Now since *CKB* is observed to be 45 degrees, its half *MKC* is $22\frac{1}{2}$; and the half of this, that is *KOC* is $11\frac{1}{4}$. Therefore in the triangle *OMF*, right angled at *F*, the ratio of the sides is given, and taking the radius *OM* to consist of 100000 parts, *MF* the sine of $11\frac{1}{4}$ degrees will be 19509 and *OF* 98078 such parts. The ratio of *OM* to *OD* is the same as of the sine of the angle of refraction *MDO* to the sine of the angle of incidence *MDL* or *DMN** that is as 187 to 250. Therefore since *OM* contains 100000 parts, *OD* will contain 133690, from whence taking *OF* 98078, there remains *FD* 35612; but *MF* was 19509. Therefore the proportion of the sides of the right angled triangle *MFD* being known, the ratio of either of them to the base *MD* will be given, which *MD* will be found 40605. Therefore *MD* the semidiameter of the globule, must be to *MF* the semidiameter of the kernel of snow, as 40605 to 19509 or as 1000 to 480 very nearly, to make the diameter of the corona 45 degrees. By the same method, to make the diameter of a corona 90 degrees, we shall find the ratio *MD* to *MF* must be as 1000 to 680; and as 1000 to 473 if the diameter be 44 degrees.

V. The diameter of a corona considered. Fig. 485. Fig. 486.

* Art. 221.

532. Concerning the formation of this sort of globules, it is probable that at first they were globules of soft snow, which are rounded by a continual

VI. The formation of the globules considered.

tinual agitation in the air, and thawed on their outsides by the heat of the sun. The appearance of corona's requires an exact temperature of heat and cold in the air, that the globules of snow may be gradually thawed a little, and be checked again by a small increase of cold. But when they are thawed about half way through their diameters, or a little more, there is a cone of rays reflected from their backsides inwards, whose heat will prevent their being congealed again, though a greater degree of cold should follow. Now by the warmth of this cone which is greatest at its apex, the snow in the center will be rounded and polished, because the globule is continually turning different sides to the sun, or rather because the warmth of the cone spreads round the whole globule. And this roundness is necessary to the production of corona's. It is probable indeed that a great many globules may be unequally thawed, but if their kernels be not rounded they cannot produce any joint effect. And perhaps by this external warmth, the cold is driven into the middle of the globule (as we see it often happens in winter time) and prevents the farther dissolution of the kernel of snow. I suppose the outsides of these globules to be watery, as is most probable, because the surface of water is generally smoother and fitter to cause regular refractions than if it was congealed. Yet sometimes it may happen that the globules, so formed as described above, may be congealed again, and may still continue round and transparent enough to produce a corona. Now besides this collection or cone of rays, which melts the snow half way through, there is also another collection, which may form and preserve other kernels of snow that are larger in proportion to the whole globule; and which may produce another corona as broad again as the former; for such corona's have sometimes been observed as will appear afterwards. It is also to be observed that this collection of rays seems to be the reason why corona's are generally observed to be of such determinate diameters; nevertheless it is not impossible that corona's may appear of any diameters; since nothing hinders the globules in a temperate air, from melting very slowly, and after a certain part of them is dissolved, from keeping that proportion so long, that corona's may appear for hours together without sensible change. Hence if a certain collection of these globules, dissolved to a certain degree, and another collection of them, dissolved to another degree, shall happen to be ranged in a higher or lower region than the former, and other collections dissolved to other degrees be placed in other regions; there will appear several corona's of different diameters at the same time; and accordingly some authors tell us they have seen six such parallel corona's concentrick to the sun[a].

Fig. 487.

a *Snellius* in his Book of the Comet in 1618.

VII. Concerning parhelia and paraselenæ.

533. Having hitherto considered the causes of corona's, in the next place we are to discourse upon parhelia and paraselenæ or mock-suns and mock-moons; in which there is a greater variety of wonderful appearances, as may be perceived by the figures annexed. For besides many surprising

prising circles, there is not only a parhelion on each side of the sun, (which two were the only ones observed in former ages) but also two or three, and sometimes four more; as in the phænomenon observed by *Hevelius* in the year 1661. One may wonder how it should be, that *Aristotle*, and *Cardan*, who wrote so many ages after him, should affirm, that there never appeared above two parhelia together. Since it is not probable that six or seven parhelia should often appear together within a few years, and that the like should have never happened for so many ages before. The reason may be, that the two lateral parhelia which are always the brightest, were only taken notice of as parhelia, and that the rest which are more languid and faint were overlooked. But by more accurate observers all those lights are called parhelia which keep fixt in a certain place, though so faint in appearance as by the unskilful to be taken for little white clouds. Now though there be a great variety of parhelia observed at different times, yet there are some things common to them all. I shall therefore attempt the solution of the most remarkable phænomenon that has been observed, and mention others by the by so far as any thing occurs that is common to all; leaving what is new and extraordinary in other phænomena to be discussed afterwards. I design therefore to consider the Roman Phænomenon observed there by *Scheiner* the 20th of March 1629; which *Des Cartes* and *Gassendus* wrote upon about that time, from whose writings I take the following description.

VIII. The *Roman* Phænomenon. Fig. 488.

534. A *the place of the observer at* Rome, B *the vertex or point over his head*, C *the true sun*, AB *a vertical plane passing through the observer's eye, the true sun and the vertex* B; *which are all projected in the straight line* ACB. *About the sun* C *there appeared two concentrick rings not compleat, but diversified with colours. The lesser and inner of them* DEF, *was fuller and more perfect; and though it was open from* D *to* F, *yet these ends* D *and* F *were perpetually endeavouring to unite; sometimes they did unite and compleat the ring and then opened again. The other exterior and fainter and scarce discernable circle was* GKI; *it had a variety of colours but was very inconstant. The third circle* KLMN *was very large and all over of a white colour, such as are often seen with paraselenæ about the moon. This was an excentrick circle passing through the middle of the sun, at first entire, but towards the end of the appearance it was weak and ragged and scarce discernable from* M *towards* N. *In the common intersection of this circle and of the outward Iris* GKI, *there broke out two parhelia* N *and* K *not entirely perfect*, K *was somewhat weak, but* N *shone brighter and stronger. The brightness in the middle of them both resembled that of the sun, but towards their edges they were tinged with colours like those of the rain-bow. They were not perfectly round and even at their edges, but uneven and ragged. The parhelion* N *was a little wavering, and sent out a spiked tail* NP *of a colour somewhat fiery, which had a continual reciprocation.*

cation. The parhelia at L *and* M, *beyond the zenith* B, *were not ſo bright as the former, but rounder and white like the circle which they were placed in; they reſembled milk or clean ſilver: the parhelion* M *was almoſt quite extinct at half an hour paſt two o' clock, excepting that ſome faint remains would revive now and then; and the circle it ſelf vaniſhed in that place. The parhelion* N *diſappeared before* K *did, and while* M *became fainter* K *grew brighter and vaniſhed laſt of all.*

IX. A fuller deſcription of this Phænomenon. Oper. Tom 3. p. 652.

535. To underſtand perfectly how this appearance was, imagine the great white circle *KLMN*, that paſſed through the true ſun, to be parallel to the horizon, having its pole *B* directly over the ſpectator's head. For as to the appearance of the figure [in *Gaſſendus* * and *Des Cartes*] in which the vertex *B* is put nearer to *LM* than to *KN*; I take it to be a miſtake in the drawing. Since in all other obſervations, wherein the poſition of this white circle is deſcribed, it is always repreſented in a ſituation parallel to the horizon: although a ſmall inclination of it is not inconſiſtent with our theory. The ſpectator *A* muſt therefore be ſuppoſed to ſtand directly under the middle point *B*, in ſuch a poſition that when he looks at the ſuns *C*, *K*, *N*, he may have the other two at *L* and *M* behind his back.

X.

536. It is farther to be obſerved, that the order of the colours mentioned in the circles *DEF*, *GKN*, was the ſame as in the corona's abovementioned: namely the red next the ſun, as appears by *Scheiner*'s deſcription of the phænomenon in the year 1630 to be ſeen in the appendix. From which it appears that the diameter of the inner circle was alſo about 45 degrees, which is the uſual bigneſs of a corona: though I am apt to think that the exterior circle in the phænomenon of 1629 was leſs than the exterior circle in the phænomenon of 1630, as appears by the figure and may be proved. In this figure it is alſo to be obſerved that the tail of the mock-ſun *N* muſt be drawn within the great circle *KLMN*, as *Hevelius* rightly obſerved in the year 1661, which phænomenon may be ſeen below in the 561ſt article. But *Scheiner* has drawn it ſo, to ſhew it was extended from the ſun. The true ſun and the mock-ſuns are alſo drawn too big in compariſon to the circles, and the circles too broad for their diameters. For the diameter of the ſun ſhould be but about the 90th part of the diameter of the corona *DEF*, or of 45 degrees; and the breadth of the circle *KLM* ſhould be almoſt equal to that of the ſun. For by the explication of the phænomenon in 1630 it appears that the breadth of that circle was not equal to the ſun's diameter.

XI. Cauſed by reflections and refractions through little cylinders.

537. Having conſidered the deſcriptions of theſe wonderful appearances let us now inquire into their cauſes. Though I ſoon perceived that globules of water, or partly of water and partly of ſnow, could not produce theſe effects, yet conſidering that parhelia are always attended with corona's, I was ſatisfied that their cauſes muſt be much alike. Conſidering there

therefore what other figures hail-ſtones might poſſibly have in the air beſides a ſpherical one, I found no other ſo ſimple as that of a cylinder. And indeed I had often obſerved that ſnow conſiſted of ſeveral ſlender oblong particles mixt with thoſe of other ſhapes; and ſeeing that ſmall globules were ſufficient for the production of corona's, I imagined that a great many little cylinders, floating in the air, might produce ſome effects of the like ſort. I remembered alſo that *Des Cartes* had taken notice of certain ſmall columns or cylinders which he had ſeen upon the ground, whoſe extremities were bounded with flat ſtars conſiſting of ſix rays or points. Having therefore conſidered the poſition of theſe cylinders, and that they could not eaſily be generated in any other but an upright poſture; and that they muſt ſometimes be partially thawed by the warmth of the ſun or air as well as the round grains; I ſoon found that all the particulars of the Roman Phænomenon might be produced by theſe ſmall cylinders; and was plainly convinced of it, after I had filled a hollow cylindrical glaſs with water, and had ſuſpended a ſolid cylinder in the middle of it. For by holding it in various poſitions to the ſun and to my eye, I found the large white circle was produced by reflection of the ſun's rays from the ſides of ſuch cylinders, and the parhelia on each ſide of the ſun by a double refraction through the ſides of them, and the other parhelia partly oppoſite to the ſun by a double refraction and an intermediate reflection, in the manner to be explained hereafter.

538. But ſomething muſt be premiſed concerning the generation, poſition, figure and magnitude of theſe cylinders. As the globules which produce corona's muſt be very ſmall, perhaps ſmaller than turnepſeed, to be ſuſtained by aſcending exhalations, ſo I ſuppoſe theſe cylinders to be exceeding ſmall and ſlender for the ſame reaſon; which minuteneſs may be ſo far from hindering the perfection of their figures, that probably it may rather contribute to it. Theſe cylinders as well as the globules muſt be formed at firſt of the ſofteſt and fineſt particles of ſnow; that is out of the moſt minute and almoſt inviſible particles of a congealed cloud; (for ſnow is nothing but vapour congealed;) now ſo ſoon as a globule is formed, by a collection of theſe particles, it naturally follows that many more particles will ſoon adhere to the bottom of it, but not to its ſides. For ſince the particles of the cloud are driven upwards by a current of aſcending vapours, and ſince on the other hand the globules tend downwards by their weight or at leaſt aſcend ſlower than the vapours; it naturally follows that the particles of the cloud which ſtrike upon the bottom of a globule will ſtick to it and by degrees change it to an oblong cylindrical figure, while other particles of the cloud will eaſily ſlip by the ſides of it. And when a vaſt multitude of theſe cylinders are thus produced at ſmall diſtances from one another, it is probable enough, that the current of aſcending air and vapours, by paſſing between them, will preſerve them in the

XII. How half-thawed cylinders may be generated.

the ſame upright poſture in which they were formed: but when they are diſperſed by the wind or otherwiſe, they cannot continue upright, but will be turned into all manner of poſitions. Now when the warmth of the ſun or of the air ſhall have melted the outſides of theſe cylinders, in the ſame manner as was mentioned of the globules, a ſmaller cylinder of ſnow will remain in the middle of each of them ſurrounded with water; and after a certain part is melted, the cylinders within will become round and perfect, and will remain in this ſtate for ſome time, for the ſame reaſons that we gave for the globules. Nevertheleſs if that coat of water ſhould ſoon be frozen, it may poſſibly remain ſufficiently tranſparent and polite to tranſmit, refract and reflect the rays of the ſun in the regular manner to be deſcribed hereafter.

XIII. The large white circle how cauſed.

539. To begin the explanation of the Roman Phænomenon, firſt I ſay that the large white circle, which appeared in it, was produced by the reflection of the ſun's rays from the outſides of the upright cylinders. For when the ſun ſhines upon a vaſt number of ſuch cylinders ſuſpended in the air; it follows that a great white circle muſt appear to paſs through the ſun and to be parallel to the horizon, and to be equal in breadth to the breadth of the ſun. To make this appear, let us ſuppoſe a larger cylinder inſtead of one of thoſe ſmall ones, and conſider in what manner the rays of the ſun are reflected from it. Upon the upright cylinder $ABCD$ let a ray EF coming from the center of the ſun, be reflected into the line EG: I ſay that EF and EG are equally inclined to the horizon. For let HEK be a line drawn through E upon the ſide of the cylinder, and imagine a plane LI to touch the cylinder along the line HEK; and the ray EF will be reflected from this plane in the ſame manner as from the cylinder[a]. Now ſuppoſing another plane paſſing through the rays FE, EG to ſtand perpendicular to the plane LI, upon the line MEN, the angles of incidence and reflection FEM, GEN will be equal[b]. Imagine a ſpherical ſurface whoſe center is E to cut off equal lines EF, EM, EO and alſo EG, EN, EP, and let EO and EP be parts of the line HEK; and let FO, FM, OM be arches of great circles in that ſpherical ſurface and likewiſe GP, GN, PN. Therefore ſince the plane $FMNG$ ſtands at right angles to the plane LI, and ſince both of them paſs through E the center of the ſphere; in the ſpherical triangle FMO the angle M will be a right one, and likewiſe the angle N in the triangle GNP. But the ſide MO is equal to NP, becauſe the angle MEO is equal to NEP; and likewiſe the ſide MF is equal to the ſide GN becauſe the angles FEM, GEN are equal. Therefore the remaining ſides FO, GP are equal, and by conſequence the angle FEO is equal to GEP; whoſe complements to a right angle are alſo equal, and meaſure the inclinations of the rays FE, EG to the horizon; which was to be proved.

Fig. 489.

a Art. 19.

b Art. 8.

The

The demonſtration is the ſame when the ray *FE* is reflected from the inſide of the cylinder; by which I ſhall ſhew that the parhelia muſt appear in the great white circle.

540. Hence it appears, when the air is full of thoſe minute upright cylinders, that the rays proceeding from the ſun's center or from any other point of his diſk, will be reflected from thoſe cylinders towards the horizon in the very ſame angle in which that point of the ſun is elevated above the horizon. Becauſe by reaſon of the ſun's immenſe diſtance, every angle of incidence *FEO* upon every cylinder, whether higher or lower, is of the ſame magnitude. Conſequently the ſpectator below can ſee the ſun reflected from thoſe cylinders only, from which a line drawn to his eye makes an angle with the horizon equal to the altitude of the ſun. Therefore if lines be drawn every way from the eye of the ſpectator making equal angles with the horizon to the ſun's altitude, it is certain that all of them together will trace out a circle in the heavens parallel to the horizon. It is plain therefore that the ſpectator will perceive a white circle parallel to the horizon and paſſing through the ſun. It is alſo eaſy to be underſtood that the breadth of this circle will be equal to the breadth of the ſun. For every point of the ſun's vertical diameter, as well as his center, will illuminate a circle of cylinders of the ſame apparent height as the illuminating point. By which means the upper and under edges of the broad white circle will be parallel, and at the ſame diſtance from each other as the uppermoſt and undermoſt points of the ſun. It is alſo remarkable that while the ſun aſcends or deſcends, this circle will alſo aſcend or deſcend, and ſo will become bigger or leſs. And moreover that diverſe ſpectators, though never ſo remote from one another, will every one ſee a different circle paſſing through the ſun, in like manner as in the rainbow. But this cannot be true of that circle which *Des Cartes* propoſes to us as the cauſe of this phænomenon. For he ſuppoſes it to be a large ſolid ring of ice ſuſpended in the air, which cannot poſſibly be directly over the heads of ſeveral diſtant ſpectators. Beſides that no reaſon can be given for its paſſing through the ſun, ſometimes for two or three hours together, as we find in fact that it does. It is alſo obſervable that no thick clouds are ſeen in the air when theſe circles appear, but ſuch only as are very thin and ſcarce viſible. For we find it taken notice of in moſt of theſe obſervations, that the ſky was very clear and ſerene. Which is no wonder in our hypotheſis, conſidering that theſe minute cylinders conſtitute a very thin cloud, uniformly extended, through which the ſun and even the blue colour of the ſky may be ſeen. It happens ſometimes that ſome parts of this circle appear faint and ſcarce viſible, for no other reaſon but want of matter or a ſufficient quantity of cylinders: excepting that the part within the halo, though there be matter enough, is leſs diſcernable, and not taken notice of in ſome obſervations, by reaſon of the

neſs of the neighbouring ſun; and chiefly becauſe the white circle is much brighter juſt without the circumference of the corona, than in other parts of it. The reaſon of which will be underſtood when we explain the lateral parhelia at *K* and *N*.

XV. The lateral parhelia explained.

541. As to theſe parhelia at *K* and *N*, I ſay they are produced by the ſame upright cylinders, which produce the great white circle, by means of a double refraction of the rays of the ſun, exactly in the ſame manner in which corona's are produced by globules with ſnow in their centers. For becauſe a certain part of each of theſe cylinders is thawed, and contains a leſſer cylinder of ſnow within it, the ſun cannot be ſeen through thoſe cylinders which conſtitute the part *KN* of the white circle, but only through thoſe which lye without it, for the ſame reaſon as in corona's. From hence it alſo comes to paſs that the interval between theſe parhelia becomes ſo much greater as the internal cylinder of ſnow is thicker in proportion to the whole cylinder. But the ſun appears brighteſt of all through thoſe cylinders on the outſides of the arch *KN*, that adjoin to it, and his brightneſs decays gradually in going outwards from thence to a certain diſtance; and this is the cauſe of the tails of theſe parhelia. But ſince the colours appear moſt vivid in thoſe cylinders that are next to the outſide of the arch *KN*, for the ſame reaſon as in corona's; theſe colours make the parhelion diſtinct from the reſt of the tail, which is leſs bright and very little coloured; in like manner as the corona and the rain-bow have but a ſmall breadth, although they are terminated but on one ſide. Now theſe tails and alſo the parhelia at the head of them, always tend towards the white circle (as will appear preſently) and make it brighter ſo far as they extend. The tail *NP* of the parhelion at *N* is drawn as appearing out of this circle, but this was either a miſtake or was deſign'd to ſhew that the tail tended directly from the ſun: as *Hevelius* obſerves more accurately in the phænomenon in 1661*. Therefore though it is not related that the parhelion *K* had a tail, neverthelesſ it is plain that a certain part of the white circle was its tail, though but a dull one; becauſe this parhelion is ſaid to be fainter than the other. And it appears by *Hevelius*'s obſervations as well as my own, that all parhelia and alſo paraſelenæ or mock-moons, on each ſide of the moon, are conſtantly adorned with tails. Laſtly the extraordinary brightneſs of the parhelia, which was ſaid to rival that of the ſun [a], will be eaſily accounted for, by conſidering that every cylinder ſhines throughout its whole length; whereas the round grains in the corona or in the rain-bow emit but a little light, ſo that one cylinder may probably afford more light than ten round grains. Therefore if there be great plenty of cylinders in the air, it is no wonder that theſe images of the ſun appear ſo bright. I ſhall now demonſtrate theſe aſſertions, by an accurate conſideration of the refraction of the rays through the ſides of the cylinders. And firſt I ſhall ſhew that theſe parhelia and their tails muſt neceſſarily appear in the great white circle.

* Art. 561.

a Art. 534.

542.

542. Let $ABCD$ be one of these cylinders hanging upright in the air; and let the ray EF fall upon its watery surface, and be refracted into the cylinder along the line FG, and at G let it emerge by refraction through the watery surface in the line GH; I say this ray GH will be inclined to the horizon in the same angle as EF is inclined to it; that is in an angle equal to the sun's altitude. For let $ABCD$ be a plane parallel to the axis of the cylinder and passing through the points F, G; and it will cut the surface of the cylinder in equal angles in the lines AB, CD parallel to each other. And consequently since the ray FG is situated in this plane between the parallel lines AB, CD, it will be equally inclined to the cylindrick surface towards F and G, so as to make the angles GFC, FGA equal to one another. Hence it is too plain to need any demonstration, that the refracted ray GH tends downwards in the same angle, as in going backwards along GF it would tend upwards along FE*; that is the angles which GH and EF do make with the sides of the cylinder are equal. Imagine two planes to be drawn, one through CD and the ray EF, and the other through AB and the ray GH; and it evidently follows that these two planes are equally inclined to the intermediate plane $ABCD$ in which FG is situated and that the angles, which measure their inclinations to this middle plane, are KCB, LBC, being made by the intersections of these planes with the base of the cylinder; and consequently that these angles KCB, LBC are equal. And since we have shewn that the angles DFE, BGH are equal, it appears from hence also that the incident ray EF and the emergent ray GH after both refractions, are equally inclined to the horizon. Therefore the suns rays so transmitted through these cylinders, cannot come to the spectator's eye but from those cylinders, whose apparent altitude is the same as that of the sun; or only from those very cylinders in which the great white circle appeared by reflected rays: and consequently the two parhelia produced by such refractions must appear also in that white circle. XVI. Why the lateral parhelia are in the white circle. Fig. 490.

* Art. 11.

543. To determine at what distance from the sun the parhelia ought to appear, we must attend to that ray of the sun which, in passing through the water, touches the internal cylinder of snow. For supposing FG to be that ray, then BC will also touch the said cylinder in the plane of its base. Therefore drawing OMN in the plane of the base, through its center N and parallel to KC which lyes under the incident ray EF, let it meet the line BL in M which lyes under the emergent ray GH; and the angle BMN will be equal to the angle which measures the inclination of two vertical planes, one of which passes through the sun and the other through the parhelion; and both of them through the eye of the spectator. And consequently in fig. 488 the part CN of the white circle, intercepted tween the sun and one of the lateral parhelia, will contain the same number of degrees of its circle, as are contained in the angle BMN. Now this XVII. The parhelia appear at various distances from the sun, and why. Fig. 490.

angular distance of the lateral parhelion from the sun, is so much the greater as the internal cylinder of snow is thicker in proportion to the whole cylinder: and moreover supposing the ratio of the diameters of these cylinders to be invariable, that angular distance becomes greater, as the sun rises higher: as will appear by the following table, whose construction, which would have been too tedious here, may be seen in the appendix to this chapter.

I.	II.		III.		IV.	
Degr.	deg.	min.	deg.	min.	deg.	min.
0.	22.	00	22.	30	45.	00
5.	22.	10	22.	38	45.	26
10.	22.	38	23.	08	46.	44
15.	23.	28	24.	00	49.	04
20.	24.	42	25.	16	52.	46
25.	26.	26	27.	04	58.	24
30.	28.	48	29.	26	67.	42
35.	31.	58	32.	42	94.	22
40.	36.	18	37.	10	The parhelion can be seen no farther in this cylinder.	
45.	42.	18	43.	14		
50.	51.	00	52.	26		
55.	64.	48	66.	64		
60.	92.	34	98.	24		

The 1st column shews the sun's altitude; the 2d the angle made by two vertical planes, one passing through the sun and the other through the lateral parhelion, when the diameter of the whole cylinder is to the diameter of the internal one as 1000 to 473; the 3d column shews the same angles between the verticals when the ratio of the diameters is 1000 to 480; the 4th when that ratio is 1000 to 680. Supposing this ratio as 1000 to 714, then the said angle of the verticals will be 88 deg. 48 min. when the sun is 25 degrees high; and can be no bigger in this altitude; and this relates to the two parhelia observed by *Hevelius*[a].

[a] Art. 561.

XVIII. That several parhelia with tails may appear together.

544. If a sufficient number of cylinders be placed above one another some of which are melted more and some less, it appears by these tables, that besides the collateral parhelia next the sun, two or more may appear farther from him, but still in the same white circle: which is also confirmed by the observations of *Hevelius* in the year 1661. Feb. 20. and of *Scheiner* in the year 1630; of which more hereafter. It appears also from the same tables, that while the cylinders remain in the same situation, and the sun rises higher, the intervals between him and the two parhelia will increase, and will decrease again while he descends lower: the very thing that I observed and shall mention in my account of that observation. But a greater alteration of these intervals may happen if the cylinders be farther melted: and from hence there arose a very remarkable phænomenon in former days. For *Julius Obsequens* relates *that in the time of Augustus Cæsar, when M. Lepidus and Munatius Plancus were Consuls, there appeared three sun's together, which soon after were contracted into one orb.* This is such an appearance as I never met with any where else; but the cause of it is easily understood, and is nothing else but the warmth of the air; which increasing upon the former temperature of it might presently melt

melt all the ſnow in the cylinders, and by this means both the parhelia would draw nearer to the ſun till they coincided with it, when the whole cylinders were melted and changed into round drops. And ſeeing this agrees ſo exactly with our theory, it cannot be doubted but we have taken the words of *Obſequens* in their true ſenſe; although *Gaſſendus* has endeavoured at another interpretation [a]; as if the parhelia appearing at firſt, there preſently broke out a corona which contained them within it. For then it would not be ſo proper to ſay that three ſuns were contracted into one orb, but two rather; neither would this have been ſo uncommon a ſight. Beſides this, they imagined that a repreſentation of the Triumvirate was ſignified by this prodigy; as appears by the following paſſage of *Dion Caſſius* compared with that of *Obſequens*. *Dion* ſays, book 45, that the light of the ſun appeared ſometimes to be diminiſhed and extinguiſhed, and ſometimes to break out again in a triple orb; (for ſo I interpret his words τότε δ' ἐν τρισὶ κύκλοις φαντάζεσθαι) one of which was ſurrounded by a radiating fiery crown; by which future events were foretold. We might accurately determine the lengths of the tails of theſe parhelia; but the farther they recede from the parhelia the fainter they grow, and therefore the true extent of them cannot well be obſerved. It is therefore ſufficient to know that they are extended throughout a whole quadrant of the white circle beginning from the true ſun; and ſomewhat farther, as the ſun's altitude may happen to be greater. But yet their light is frequently ſo weak as by obſervers to be taken only for a part of the ſaid circle.

a Op. Tom. 3. p. 655.

XIX. The corona paſſing thro' the parhelia is not made by globules.

545. I am now going to conſider thoſe corona's which almoſt always appear with theſe lateral parhelia. For though corona's are ſometimes ſeen without parhelia, yet parhelia never appear without a corona paſſing through them, unleſs it be ſo weak ſometimes as to eſcape obſervation; and this appearance being not accidental muſt proceed from a certain cauſe. When ſmall cylinders are produced in the air, it is probable that half-thawed globules are lodged there at the ſame time, for the production of corona's. But yet it is difficult to conceive by what cauſe they can be thawed in ſo exact a proportion, as to produce a corona that ſhall paſs preciſely through theſe very parhelia. For ſuppoſing the round and oblong hail to be thawed exactly alike in diameter, nevertheleſs it appears by the foregoing tables that the parhelia would lye on the outſide of the corona. For when the ratio of the diameter of the outward to that of the inward cylinder is as 1000 to 473, the diſtance of the parhelia from the ſun at various altitudes will be 22, 28, 36, 51, and any more degrees of the white circle, though the ſemidiameter of the corona be but 22 at all altitudes of the ſun, when the globules are thawed in the ſame ratio. This alſo is farther to be obſerved that while the parhelia continually approach towards each other when the ſun deſcends, and recede from each other when he aſcends, as we obſerved above; the corona ſhould alſo contract and

and enlarge it ſelf, ſince it always paſſes through the parhelia; but in the round grains no cauſe can be aſſigned for ſuch an effect.

XX. Nor by cylinders obliquely ſituated.

546. But beſides the round grains, it is certain that thoſe very cylinders, which I obſerved may ſwim in the air in uncertain poſitions, are capable of producing corona's. For by reaſon of the ſnow within them they do not tranſmit the rays of the ſun, unleſs they lye wide of a certain diſtance from him; which angular diſtance is determined by thoſe cylinders on whoſe ſides the rays of the ſun fall perpendicular; for the reſt of them upon which the rays fall obliquely muſt be ſtill remoter from the ſun to tranſmit his rays to the eye: as appears by the tables above, taking the altitude of the ſun in the table for his altitude above the plane of the baſe of theſe cylinders. So that theſe cylinders as well as the globules may produce a corona bounded with a red colour on its ſide next the ſun. And probably the corona's that appear with the parhelia may proceed from this cauſe. Firſt becauſe theſe little cylinders are alone ſufficient. Secondly becauſe the corona's that appear with parhelia are often brighter coloured than without parhelia. Becauſe the cylinders ſend rays from their whole lengths in greater plenty than the round grains, which ſend them as it were but from a point. But to return to the purpoſe, there is no reaſon at all why this corona produced by cylinders ſhould paſs through the parhelia rather than that which proceeds from round grains. And the ſame difficulty in the alteration of the diſtance of the parhelia ſtill remains as before. There is no reaſon therefore to maintain that the corona which paſſes through the parhelia is produced by thoſe cylinders that are placed in an uncertain order. Indeed as to the Roman Phænomenon fig. 488 I do not doubt but the corona *DEF*, within the parhelia, was produced by cylinders diſpoſed in the manner abovementioned. But ſtill as to the other part of the phænomenon there ſeems to be ſome difficulty.

XXI. But by the round ends of upright cylinders.

547. Nevertheleſs we may be able to conquer this difficulty by attending more cloſely to the figure of thoſe cylinders. Hitherto we have only conſidered their ſides; now let us ſee what ſort of extremities they are likely to have. To ſuppoſe them terminated by plane baſes, is by no means agreeable to nature. But becauſe their tops and bottoms are thawed as well as their ſides, and to the ſame degree quite round, moſt certainly the water at their extremities will affect a round figure; whence alſo the inner cylinder of ſnow will be formed at each end either into an hemiſphere or an half ſpheroid; and then the ſhape of the whole cylinder with the included kernel ought to be ſuch as this figure repreſents in a larger ſize.

Fig. 491.

Hence it will come to paſs, that not only thoſe upright cylinders, which lye on each ſide of the ſun and produce the collateral parhelia, will tranſmit his rays to the eye, but alſo thoſe that lye above and below and quite round him beyond a certain diſtance; which therefore will produce a corona. Through thoſe above him the rays will paſs as the ray *DCBA* does and

and will touch the top of the kernel in *H*; and the rays will paſs in like manner through the bottom of the lower cylinders as *EFGA* does, and will touch the bottom of the kernel in *K*. But ſince the exact figure of the convexities *BCL* and *FGM* is unknown, it would be in vain to attempt an exact determination of the quantity of refraction of the rays that paſs through it; and much more of thoſe that paſs towards the ſides of theſe oval ſolids. Nevertheleſs having ordered a glaſs cylinder to be made in the ſhape repreſented by half this figure; and having filled it with water and ſuſpended an opake cylinder within it, that might in all places be equally diſtant from the glaſs; and having placed this compound cylinder in various poſitions between the ſun and my eye, I found by experience it tranſmitted the rays in ſuch manner that the angles *BAN*, *GAF* were nearly equal to one another and to all other angles under the rays that touched the opake cylinder in paſſing by it; taking the line *AN* for a line drawn from the eye to the ſun; or for a line parallel to it.

548. Now ſince the little cylinders in the clouds muſt have the ſame power and property as this fictitious one, they will form a corona which will be nearly circular. So that theſe upright cylinders, which formed the great white circle by reflection, will alſo form thoſe corona's by refraction that paſs through the parhelia. Conſidering therefore the neceſſity we have ſhewn for the ſhape of theſe cylinders, and that the rays paſſing through their extremities will produce coloured circles, I do not doubt but theſe circles are thoſe very corona's; otherwiſe ſome irregular and ill-ſhaped circles muſt neceſſarily be viſible. And though the eye may not be able to diſcern a ſmall defect of roundneſs, yet it may happen that one diameter of a corona may be different from another. Which I find is confirmed by obſervation. For in an obſervation made at *Rome* in the year 1630, when two circles about the ſun are ſaid to croſs each other above and below him; and that parhelia ſhone out from the outward arches on each ſide of him; it ſeems more probable to me that inſtead of two circles croſſing each other, we muſt firſt conceive a circle pretty oblong, in which two parhelia were produced by the upright cylinders, and then another circle not quite ſo oblong, touching the other above and below, produced either by cylinders in a confuſed ſituation or by round grains. For the figure made by theſe oblong circles is ſo very like the figure made by two perfect circles, as not eaſily to be diſtinguiſhed from it. And beſides it is exceedingly probable that one and the ſame circle contained two parhelia. XXII. Fig. 492.

549. The circle that paſſes through the parhelia frequently appears fainter above and below, and brighteſt next the parhelia; as *Hevelius* has accurately remarked in many obſervations. The reaſon of this is alſo plain by our glaſs cylinder. For the rays paſſing through the round tops and bottoms of them produce only a ſmall round image of the ſun, as round grains

XXIII. The parts of this corona are brighteſt next the parhelia.

grains do; but those which pass towards the ends of the cylindrical surfaces where they just begin to turn roundish are formed into a broader and brighter pencil; in the same manner as those are which produce the parhelia; which makes the corona grow gradually brighter and brighter towards the parhelia both above and below, and more intensely coloured.

XXIV. The back parhelia *L*, *M* how produced.

550. Having explained the causes of the two corona's in the Roman Phænomenon, let us proceed to account for those two parhelia *L*, *M* which appeared in the back part of the great white circle. These I say, are produced also by the upright cylinders; by the same refractions of the sun's rays which produce the rain-bow. It must be observed and shall be demonstrated by and by, that the inner opake cylinders are so far from contributing to the production of these parhelia, that sometimes they hinder their appearance. But yet the inner cylinder of snow is absolutely necessary, whether the outward shell that contains it be made of water or ice. For without the snow the watery cylinders would immediately be changed to round drops; and the whole cylinders cannot be all ice, because we cannot conceive them to be every where transparent, without supposing them quite liquid before they were frozen; which I have just now shewn to be inconsistent with a cylindrical figure.

XXV. The bases of the rain-bow answer to two back parhelia when the sun is in the horizon. Fig. 493. a See Art. 494. b See the remarks upon chap. 10.

551. But before we consider the manner of refraction through cylinders, it will be convenient first to consider it in round drops, to shew the cause of the rain-bow, which *Des Cartes* first [a] discovered. Let *ABC* be a globule of water, *DA* a ray of the sun falling upon it, and refracted from *A* to *B*, and then reflected from *B* to *C*; so that *AB* and *BC* may be equal; and at *C* let it be refracted again towards the eye at *E*. Again drawing *CH* and *EF* parallel to the ray *DA*, and making the angle *FEG* equal to *FEC*, it appears by *Des Cartes*'s tables [b], that the angle *HCE* will vary its magnitude according as the ray *DA* is supposed to fall upon different parts of the drop; but so as never to exceed $41° 30'$; and therefore the angles *CEF* and *FEG* never exceed that quantity. Hence it follows that no drops that lye out of the cone *CEG* can transmit any rays to the eye; and of those that lye within it, the nearest to the conick surface will transmit them most plentifully; and by this means the brighter drops which boarder upon the dark ones will be more apparent and will make a round ring; which would be visible quite round but for the surface of the earth; whereas one half at most can only be seen when the sun is in the horizon. This is the true cause of the rain-bow, which any one may examine experimentally, by exposing a glass globe full of water in the same position to the sun as the globule *ABC*. Suppose now the circle *ABC* to be an horizontal section of an upright cylinder, and that innumerable others are placed upright upon the plane of the horizon, and that the rays of the sun being also in the horizon are refracted through them in the same manner as through the drop *ABC*, so as to make the greatest

angle

angle HCE or CEF equal to $41^\circ\ 30'$: and it follows that no cylinder out of the angle CEG can transmit a ray to the eye; and that those which are nearest to the surface aforesaid will transmit the most rays. Therefore a small portion, like the base, of a rain-bow will appear in the cylinders towards C, and another likewise towards G; and the intercepted arch of the horizon will contain as many degrees as the angle CEG does, that is 83°. Thus we have two parhelia much brighter than the parts of the rain-bow, as being produced by cylinders, which transmit more rays than globules do, as we have often observed before. Thus much for the appearance of parhelia when the sun is in the horizon.

552. But the sun being elevated, these two back-parhelia will appear in the white circle. To demonstrate this, let $ABCD$ be an upright cylinder in the air; EF a ray of the sun falling upon it; FG the same ray refracted into the cylinder, GH the same ray reflected from its surface at G, and HK the same ray emerging by refraction at H: I say the incident and emergent rays EF, GH will be equally inclined to the plane of the horizon. XXVI. Why the back parhelia appear in the large white circle. Fig. 494.

553. For by the laws of reflection the ray FG will be so reflected into the line GH, that two planes drawn through FG, GH parallel to the axis of the cylinder, will cut the bases and sides of the cylinder in two parallelograms $PAQC$, $ABDQ$ equal to each other in length and breadth; and the rays FG, GH will be equally inclined to the sides of these parallelograms. This is plain enough by what has been demonstrated and observed at the end of the 539th art. Therefore since the rays FG, GH are equally inclined to the lines PC, BD, the ray GH in going forward will emerge by refraction at H along HK, and GF supposed to go backwards will emerge at F along FE* in such manner that the angles PFE, DHK will be equal. Now let the line CL lye in the plane of the base directly under EF, and also DM under HK; and the angles QCL, QDM will be equal; and since the angles PFE, DHK are equal, the rays EF and HK will be equally inclined to CL and DM, that is to the plane of the horizon. Therefore the sun's rays transmitted along these lines that bend about the cylinders hanging in the air, cannot come to the spectator's eye, but from those cylinders only whose apparent height above the eye is equal to the apparent height of the sun; that is from those only which caused the white circle to appear by reflections from them. And so it is manifest that the parhelia thus produced can appear no where else but in the said white circle. XXVII. Demonstration.

* Art. 9.

554. Now the reason why these parhelia appear in certain determinate places of the white circle, is altogether similar to that which we gave for the rain-bow. For if in the plane of the base the lines DN, MO be drawn parallel to CL which lyes under the incident ray EF; I find that the angle MDN or DMO cannot exceed a certain determinate magnitude; and that its greatest magnitude will vary while the sun's altitude varies. XXVIII. The distance between the back parhelia determined. Fig. 494.

For example, when the ſun is 25 degrees high this angle is 33° 18′ at moſt; and ſince the rays which fall upon the cylinder emerge in the limit of this greateſt angle more copiouſly than when it is ſmaller; this greater quantity of rays is the cauſe that the parhelia appear on each ſide of *MO* at that angular diſtance. Drawing therefore *MR* in the plane of the baſe ſo as to make the angle *OMR* equal to *OMD*, the angular diſtance of the two parhelia will be equal to the whole angle *DMR*: ſo that two vertical planes ſtanding upon the lines *MD*, *MR* will paſs through the parhelia; and the arch of the horizontal circle and likewiſe that of the white circle parallel to it and intercepted between theſe planes, will contain exactly as many degrees as the angle *DMR* does. The number of degrees in half this angular diſtance is computed in the following table for ſeveral altitudes of the ſun. But the manner of making the computation, being too prolix for this place, is taught in the appendix to this chapter.

Degr.	deg.	min.
0.	41.	30
5.	41.	08
10.	40.	14
15.	38.	36
20.	36.	16
25.	33.	18
30.	29.	36
35.	25.	16
40.	20.	12
45.	14.	40
50.	8.	44
55.	3.	06
58.	0.	32

The ſecond column of this table ſhews the angle between a vertical plane, paſſing through the ſun, and another paſſing through one of the parhelia, anſwering to the ſun's altitude put over againſt it in the firſt column.

XXIX. The deſcription of the Roman Phænomenon corrected.

555. By this table I find the diſtance between the back parhelia of the Roman Phænomenon to be about 60 degrees. For the height of the pole at *Rome* being 42° 2′, the ſun's altitude there at three o' clock in the afternoon was about 30 degrees; and in this table half the diſtance of the parhelia anſwering to this altitude is 29° 36′. Indeed in *Scheiner*'s ſcheme as it is drawn in *Gaſſendus*'s little book [a], the diſtance of theſe parhelia exceeds 90 degrees; but it was neither meaſured nor eſtimated, nor ſo much as mentioned; and therefore I dare affirm it is repreſented too big in the figure. For the diſtance between any two points in the heavens appears ſo much the greater as the points are nearer to the horizon: juſt as the diſtances between the ſtars in the great Bear, for example, ſeem to be twice as big when they are near the horizon as when they are near the vertex. For the like reaſon the diſtance between theſe two parhelia ſeemed to bear a greater proportion than it really had, to an arch, which paſſing through the vertex, defined the magnitude of the diameter of the white circle *KLMN*. But this could not happen in the diſtance between the parhelia *K*, *N*, becauſe of the known magnitude of the circle *DEF*, whoſe diameter is 45 degrees. The ſame fallacy in the diſtance of the back parhelia will appear plainer in ſome of the following obſervations. It is alſo upon the ſame account, that the diſk of the ſun appears almoſt twice as big in the horizon

[a] See Oper. Tom. 3. p. 652.

Fig. 488.

as

as when he is higher elevated; and likewiſe that the rain-bow appears to be a part of the greateſt circle of a ſphere, whereas it is not half ſo big.

556. Now the cauſe of this fallacy in ſhort is this; that we think the ſun or any thing elſe in the heavens to be remoter from us when it is near the horizon, than when it approaches towards the vertex. Becauſe we imagine every thing in the air, that appears near the vertex, to be no farther from us than the clouds that fly over our heads. Whereas on the other hand, we are uſed to obſerve a large extent of land, lying between us and the objects near the horizon, at the far end of which the convexity of the ſky begins to appear; which therefore together with the objects that appear in it, is uſually imagined to be much farther from us. Now when two objects of equal magnitudes appear under the ſame viſual angle, we always judge that object to be larger which we think is remoter. And this is the true cauſe of the deception we have been ſpeaking of.

XXX. Why objects in the sky appear larger near the horizon than higher up.

557. But to return to the parhelia; it muſt be obſerved, that in the Roman phænomenon they are ſaid to have appeared white, whereas they ought to have been coloured, for the ſame reaſon as in the rain-bow and in the glaſs cylinder filled with water. I take the cauſe of this whiteneſs to ariſe from the weakneſs of their light, as we ſaid before of corona's that ſometimes appear white [a]. This alſo happens to the lateral parhelia, as appears by the accurate obſervation of *Sam. Kechelius* to be ſeen in the appendix: in which the parhelion *C*, which was fainter than the other, is ſaid to change from yellow to white, which is the ſame as the ſilver colour obſerved in our parhelia. Now the yellow and white colours, which the cylinders tranſmit by the refractions aforeſaid, being far more ſplendid than the red, it muſt neceſſarily follow, that when but few cylinders remain, the red will diſappear before the yellow and white; and ſo the light of the parhelia will remain without any tincture of colours. But that theſe back parhelia do ſometimes appear coloured, is evident from the Engliſh obſervation, taken from *Math. Paris*'s Hiſtory, to be ſeen in the appendix. Where we are told that beſides the true ſun, four mock-ſuns appeared red in a great circle of the colour of chryſtal; and that two of theſe were back parhelia, is manifeſt from the figure, though faulty perhaps in other reſpects.

XXXI. The colours of the back parhelia conſidered.

a Art. 530.

Fig. 503.

Fig. 502.

558. This ſcarcity of cylinders may perhaps be the reaſon that in many obſervations theſe parhelia have not appeared, although the white circle was plain enough to be ſeen: as in a phænomenon at *Rome* in the year 1630 and in that ſeen by *Hevelius* in the year 1661 Feb. 20 [b]; and in ſome others: in which the parhelia did not appear by reaſon of too faint a light. Nevertheleſs there is another thing which may hinder their appearance; namely if in the back part of the white circle, the inner cylinders contained in the water be too thick in proportion to the water; for then they will ſtop the rays which ſhould produce theſe parhelia. For I find when the

XXXII. Why theſe parhelia don't appear ſometimes.

b Art. 561.

 ſun

ſun is 25 degrees high, if the diameter of the opake cylinder ſhall bear a greater proportion to that of the whole than 590 to 1000, that no back parhelia can appear. Hence in a phænomenon obſerved at *Rome* in the year 1630, when the ſun was 28 degrees high, if the cylinders that produced the parhelia *M* and *N* did not extend ſo far as to the back part of the white circle, and no others lay there but ſuch as produced the parhelia *O* and *P*, no back parhelia could appear in that region: ſince in theſe cylinders the ratio of the internal to the external diameter was as 624 to 1000. Likewiſe in *Hevelius*'s obſervation when 6 parhelia appeared, the ſun being 25 degrees high, if the cylinders that produced the lateral parhelia *B* and *C* did not extend to the back part of the white circle, and no others lay there but ſuch as produced the parhelia *E* and *D*, the back parhelia could not appear. For the ratio of the diameters of theſe internal and external cylinders was 714 to 1000.

Fig. 498.

Fig. 495.

XXXIII. Why the back parhelia have no tails. Fig. 488.

* Art. 551. 552.

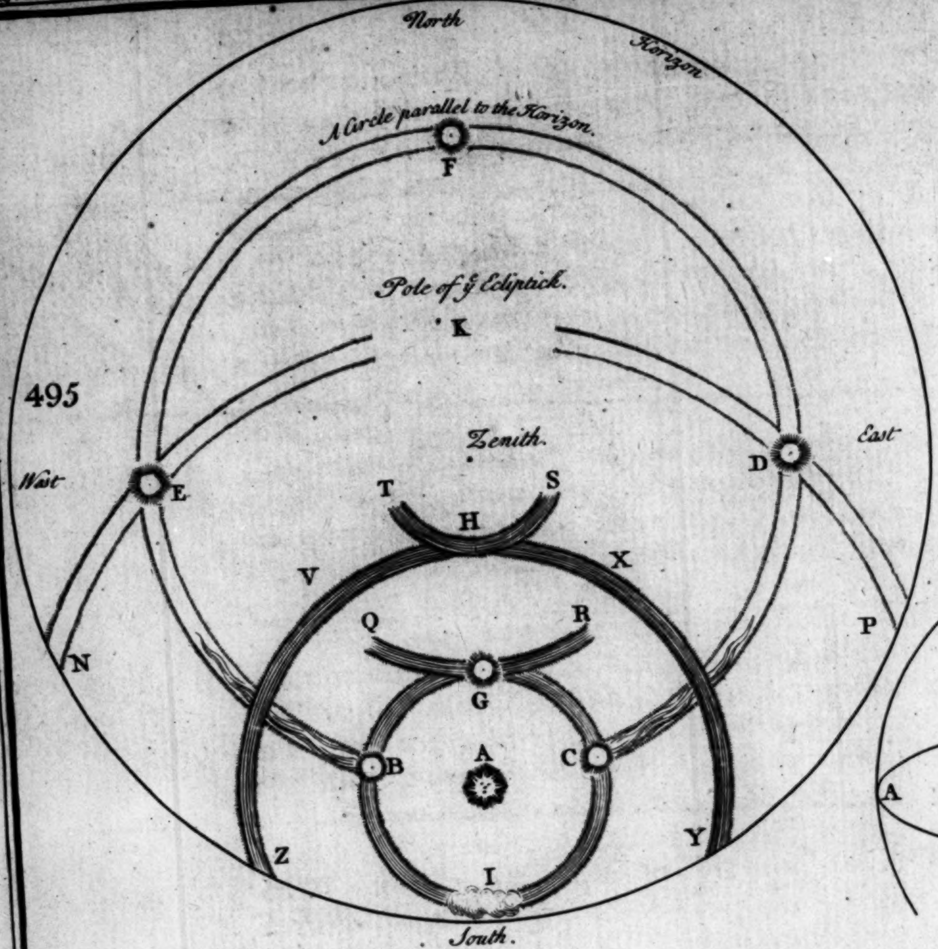
559. The reaſon why the back parhelia in the Roman Phænomenon appeared rounder than the lateral ones is this. Although ſome cylinders in the arch *LM*, that lye not far from *L* and *M*, ſend ſome refracted rays to the eye, yet they are not near ſo numerous as thoſe which come from the cylinders at *L* and *M**; and therefore theſe parhelia have no tails that are far extended, like the lateral ones. Beſides this, the refractions through the cylinders that produce the back parhelia, are more regular. For the rays are not confined in their courſe by the opake cylinders within, but receive their directions from the refractions at the moſt exact figure and poliſh of the outward watery ſurface of the cylinders. For the dark cylinders within not being exactly equal to one another, are the cauſe of the irregular and inconſtant ſhape of the lateral parhelia. The reaſon of the reciprocation or fluctuating motion of the tail of the parhelion *N*, was that ſometimes fewer and ſometimes more cylinders were carried into that place; which is alſo the reaſon that the corona *DEF* was ſometimes entire and ſometimes open below, and that the parhelion *K* grew brighter while *N* grew duller.

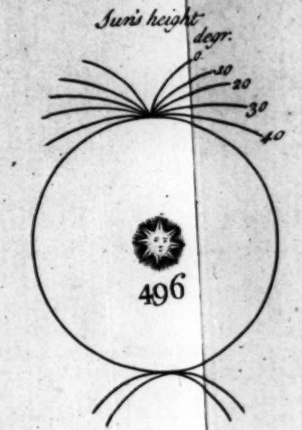
XXXIV.

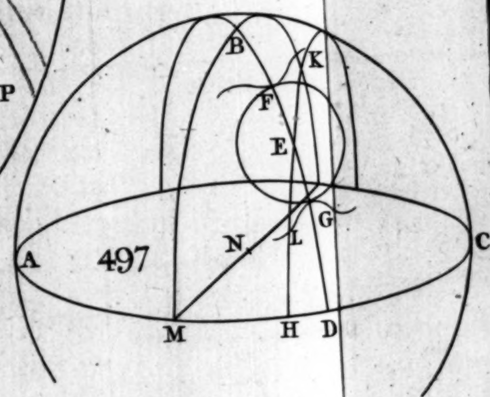
560. I have now gone through the explication of every particular of the Roman Phænomenon, referring the cauſes of all the parhelia and circles in it to half-melted cylinders, partly upright and partly in a confuſed order; which cauſes, ſo agreeable to the phænomena and alſo to one another, do every where confirm one another; ſo that I think the reality of them cannot be doubted.

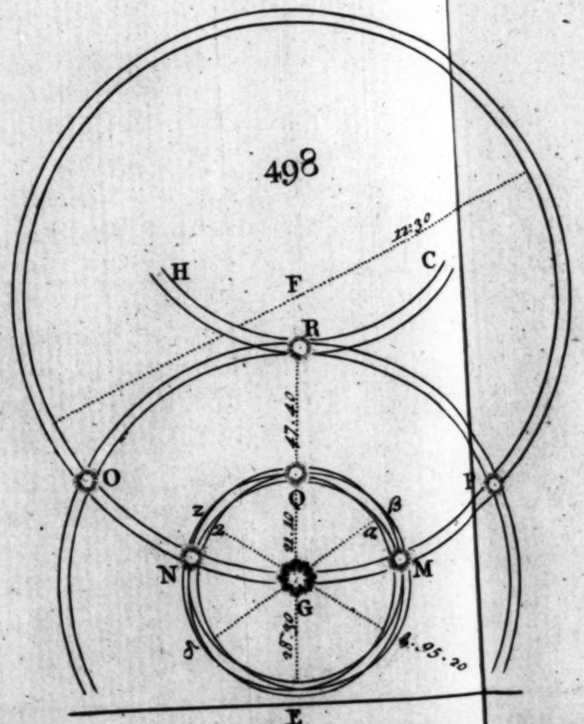
XXXV. Tranſition to an extraordinary phænomenon of 7 ſuns ſeen at *Dantzick* by *Hevelius*.

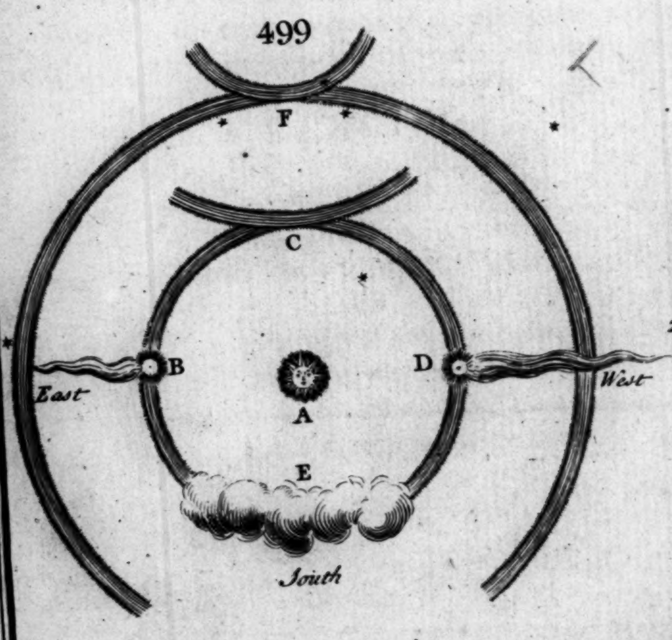
561. Having accounted for every particular in the Roman Phænomenon, let us now proceed to *Hevelius*'s obſervation, 20th Feb. 1661; in which more ſuns and circles were obſerved, and ſome of them in a different poſition from thoſe we have hitherto conſidered. But ſince the cauſe of them is not to be found either in the upright cylinders, or in thoſe that fly about confuſedly; (for our glaſs cylinder viewed in all poſitions diſcovers


North
Horizon
A Circle parallel to the Horizon.
F
Pole of ÿ Ecliptick.
K
495
Zenith.
East
West
D
E
T
S
H
V
X
Q
R
N
P
G
B
A
C
Z
Y
I
South.


Sun's height
degr.
0
10
20
30
40
496


B
K
F
E
G
L
N
A
497
C
M
H
D


498
12.30
H
F
C
R
47.40
O
Q
P
z
β
21.10
N
M
G
28.30
45.20
E


499
F
C
B
D
East
West
A
E
South

covers nothing elſe but the particulars of the Roman Phænomenon;) it ſhould follow that ſomething elſe muſt happen in the air: and yet I ſhall ſhew it to be nothing more than a certain poſition of thoſe cylinders, which has not yet been conſidered. But in the firſt place let us repreſent the phænomenon it ſelf, by *Hevelius*'s figure deſcribed in his own words, taken from the appendix to his *Mercurius in Sole viſus.* pag. 174.

On Sunday in the year of our Lord 1661. *Feb.* 20. *New Style, a little before eleven o' clock, the ſun being towards the ſouth and the sky very clear, there appeared ſeven ſuns together in ſeveral circles, ſome white and ſome coloured, and theſe with very long tails, waving and pointing from the true ſun, together with certain white arches croſſing one another.* 1. *The true ſun at* A, *being about* 25 *degrees, high was ſurrounded almoſt entirely by a circle whoſe diameter was* 45 *degrees, and which was coloured like the rain-bow with purple, red and yellow; its under limb being ſcarce* $2\frac{1}{2}$ *degrees above the horizon.* 2. *On each ſide of the ſun, at* B *and* C *towards the weſt and eaſt there appeared two mock-ſuns, coloured, eſpecially towards the ſun, with very long ſplendid tails, of a whitiſh colour and terminating in a point.* 3. *A far greater circle* YXHVZ *almoſt* 90 *degrees in diameter encompaſſed the ſun and the former leſſer circle* GBIC, *and extended it ſelf down to the horizon. It was very ſtrongly coloured in its upper part, but was ſomewhat duller and fainter on each ſide.* 4. *At the tops of theſe two circles at* G *and* H *were two inverted arches, whoſe common center lay in the zenith, and theſe were very bright, and beautifully coloured. The diameter of the lower arch* QGR *was* 90 *degr. and that of the upper one* THS *was* 45 *degr. In the middle of the lower arch at* G, *where it coincided with the circle* BGC, *there appeared another mock-ſun, but its light and colours were dull and faintiſh.* 5. *There appeared a circle* BEFDC *much bigger than the former, of an uniform whitiſh colour, parallel to the horizon at the diſtance of* 25 *degr. and* 130 *degr. in diameter; which aroſe as it were from the collateral mock-ſuns* B *and* C, *and paſſed through* 3 *other parhelia of an uniform whitiſh colour like ſilver; one at* D *almoſt* 90 *degr. from the true ſun towards the eaſt; another at* E *towards the weſt; and a third at* F *in the north, diametrically oppoſite to the true ſun; all of the ſame colour and brightneſs. There paſſed alſo two other white arches* EN, DP *of the greateſt circle of the ſphere, through the eaſtern and weſtern mock-ſuns* E, D, *and alſo through* K *the pole of the ecliptick; they went down to the horizon at* N *and* P, *croſſing the great white circle obliquely, ſo as to make a white croſs at each parhelion. So that ſeven ſuns appeared very plain at the ſame time; and if I could have ſeen the phænomenon ſooner from an eminency, I do not queſtion but I ſhould have found two more at* H *and* I, *which would have made nine in all. For there remained in thoſe places ſuch marks as made this ſuſpicion not improbable.* Fig. 495.

This

This most delightful and extraordinary sight lasted from 30 *minutes past* 10 *to* 51 *min. past* 11; *though it had not the same appearance all that while, but sometimes one and sometimes another. It appeared in the perfection of this description at about eleven a clock, and then degenerated by degrees. The northern mock-sun at* F *vanished first of all together with a part if its circle; the other parhelia with their arches lasted till ten minutes past eleven, then the eastern mock-sun and after that the western vanished with both the crosses. Soon after this the collateral parhelia* C, D *suffered several changes, sometimes one was brighter than the other in light and colours, and sometimes fainter and darker. For at* 18 *min. past* 10, *the eastern parhelion* C *vanished while the western parhelion at* B *remained very conspicuous; and* 24 *min. past* 11 *the eastern one was very bright again and remained so while the western one disappeared at* 40 *min. past* 11; *although this western one had almost always the longer tail. For the tip of it was frequently extended for* 30 *degr. and sometimes* 90, *as far as the parhelion* E; *but the tail of the eastern one* C *was scarce above* 20 *degrees. At* 30 *min. past* 11 *the great vertical circle* YXHVZ *was destroyed; but the inverted arches* H *and* G *together with the collateral parhelia* B *and* C, *continued to the last.*

The scheme of this phænomenon is drawn in the same manner as the constellations are drawn upon an artificial globe, to be viewed by the eye on the outside of it. For by this means every thing is represented much clearer and distincter. Nevertheless the place of the observer was nearly under the zenith within the circle parallel to the horizon; so that the true sun appeared to him in the meridian, the mock-sun F *in the north, and the other two at* D *and* E *on each hand. But if you desire to have this extraordinary phænomenon represented a little plainer; upon an artificial globe whose pole is elevated to our altitude at* Dantzick, *with the center* A, *in the* 2d *deg. of Pisces, where the sun then was, and with a semidiameter of* $22\frac{1}{2}$ *degrees, describe the circle* GBIC; 2. *and then the circle* YXHVZ *with a radius of* 45 *degrees;* 3. *and with the same center and semidiameter of* 90 *degrees, draw the circle* NEKDP *through the two white mock-suns* E, D. 4, *And with a semidiameter of* $22\frac{1}{2}$ *the zenith being the center, draw the arch* THS; 5. *and also the arch* QGR *with a radius of* 90 *degrees upon the same center;* 6. *and lastly the circle* BEFDC *parallel to the horizon with a radius of* 90 *degrees. And the draught being finished in this manner will appear very beautiful and harmonious; and, by the grace of God, may afford an occasion of discovering the natural causes of all sorts of parhelia and paraselenæ.*

XXXVI. A solution of several particulars of this phænomenon.

562. In this phænomenon, as well as in that at *Rome*, we have the large white circle; the collateral parhelia *B*, *C* and the corona *BGCI*; all which are produced by cylinders in an upright posture, as we have explained above; and likewise the parhelia *D*, *E*, with the arches *NE*, *PD* passing through them, for the reasons given in the xxi and xxii sections. And by what has been said in xxiii and xxxiv sections it appears why this circle *NEKDP*,

NEKDP, which has the ſun for its center, had a little opening at the top of it: and in the xxxii ſection it is explained how it happened that no back parhelia appeared. The diſtance of the two parhelia *E* and *D* from the ſun *A*, ought to have been within 88° 48′. For when the ſun was 25 degrees high the lateral parhelia could not appear at a greater diſtance from him[a]; and being at this limit, they ought to appear very faint and languid in compariſon to the two parhelia *B* and *C*. For this reaſon they appeared white as well as the arches *EN*, *DP* according to ſect. xxxi. The circle *ZHY* was produced either by round grains or by cylinders flying about confuſedly according to ſect. xx; or laſtly for another reaſon to be mentioned in the xli ſection.

[a] Art. 543.

XXXVII. A great many cylinders lye horizontal.

563. But the chief thing to be inquired into, is the cauſe of the arches *THS*, *QGR* and of the parhelia *G* and *H* that appeared in them. For the parhelion *F*, diametrically oppoſite to the true ſun, ſhall be conſidered laſt of all. We obſerved above that the cylinders were ſupported in the air by vapours aſcending from below like a wind; which contributed both to their production and to their upright ſituation. Now beſides this upright poſture I find that moſt of the cylinders ought to affect an horizontal one; and it appears by experiments that they really muſt do ſo. For if a parcel of cylinders be ſuffered to deſcend with a gentle motion either through the air or through water; they will almoſt always fall down in an horizontal poſture; that is as ſoon as they have deſcended ſo far that their celerity cannot be farther increaſed. Now if the water or air through which theſe cylinders deſcend, be ſuppoſed to tend upwards with the ſame velocity as the cylinders tended downwards; the caſe comes to the ſame as that of little cylinders ſupported in the atmoſphere: ſo that no one can doubt but a great number of them lye horizontal. Indeed from the experiments above mentioned it might be inferred, that almoſt all of them ſhould be horizontal, and ſcarce any of them upright. But it muſt be conſidered that our experiments are but an imperfect imitation of what nature does in the atmoſphere; for the upright poſture is abundantly atteſted by thoſe effects which have been clearly demonſtrated to follow from it.

XXXVIII. Hence follows thoſe arches that are convex towards the ſun.

564. I ſay then that theſe inverted and coloured arches, ſuch as *QGR*, *THS*, are produced by thoſe cylinders, whoſe axes are parallel to the plane of the horizon, though not to one another. To underſtand this matter, firſt let us ſuppoſe a vaſt number of cylinders lying promiſcuouſly, with their axes pointing towards the eaſt, ſouth-eaſt, ſouth, weſt, and all other points of the compaſs; and then let us conſider that the rays which produce theſe arches are tranſmitted through the cylinders by two refractions, in like manner as thoſe were which produced the collateral parhelia. Now ſuppoſing the ſun to be in the ſouth, for example, it will follow that thoſe parts of theſe arches, at *H* and *G*, which lye neareſt to the ſun, will be produced in thoſe cylinders whoſe axes point directly eaſtwards or weſt-

wards;

wards; that is in those cylinders which receive the sun's rays at right angles to their axes. For of all the cylinders in all positions, these that point eastward or westward are the nearest to the sun that can transmit his rays to the eye[a]. And for other cylinders to do the same thing in other positions, it is requisite they should lye farther and farther from the sun; according as their axes point farther and farther from the east or west. Because the rays of the sun are more and more elevated above the planes of their bases; and consequently produce the several parts of these inverted arches, which lye farther and farther from their middle parts at *G* and *H*. Having carefully considered what sort of shapes these arches will have[b], I find this difference: *viz.* when the sun is in the horizon the inverted arches, one of which touches the circle of 45 degrees and the other that of 90 degrees, are like the horns in the 496th figure. But the lesser of the two horns shews the shape [of the arch[c]]; and the red colour is always next the sun. The same figure shews also the said arches when the height of the sun is 10, 20, 30, 40, degrees. From which it appears that the middle of these horns are like pieces of circles, but towards their extremities they are bent the contrary way. And since the sun shines more directly upon those cylinders which produce the middle of the arches, and more obliquely upon those which produce their extremities, it is no wonder that only the middle parts are visible, as being brighter than the rest.

a Art. 543.

b See Art. 567.

c minor autem duorum cornuum figuram exhibet.

XXXIX. The parhelia in the inverted arches explained. Fig. 495.

565. Now the parhelia which appear exactly in the middle of these arches, as at *G*, are nothing else but the brightest parts of them; and therefore never appear quite distinct nor much brighter than the adjoining parts of the arches. Accordingly *Hevelius* in all his observations of this kind, that is, of two parhelia and one paraselenæ, has taken notice that they appeared somewhat dull, heavy and faint. And he was not sure whether there was a parhelion at *H* or not. Now the reason why the middle part of the arch appears a little distinct from the rest may be this. That though there be great plenty of horizontal cylinders in the air, yet they may be so short as to have but little of a cylindrical surface, or perhaps may be like oblong spheroids. Besides we find it perfectly agreeable to *Hevelius*'s observations, that the higher the sun or moon and the corona are elevated above the horizon, the inverted arches appear the flatter. For in his observation of the paraselenæ, in which the moon's altitude was 26 or 27 degrees, the arch over the first corona was part of a very large circle; as likewise in the phænomenon of 7 suns fig. 495. But in that of Apr. 6. and Decemb. 17. 1660, in which the sun's altitude was but 12 or 13 degrees, the inverted arches were parts of much smaller circles. It must be confessed that the upper parts of these arches are represented much less by *Hevelius* than according to the following calculation. But the reason is, that arches so highly elevated above the horizon do necessarily deceive the sight; so as to appear to be parts of much lesser circles than they really are. For by

Fig. 499 to 501.

what

what has been ſaid in the xxix ſection, it is evident that the ſame circle overhead appears but half as big as when it is near the horizon, which is true likewiſe of the parts of arches.

566. As to the arches which according to our hypotheſis ought to adhere to the loweſt part of the interior corona; in *Hevelius*'s two obſervations upon Mar. 30. 1660. fig. 499. and in that of the 7 ſuns, theſe arches might have been obſerved; and indeed they did appear in ſome meaſure, eſpecially in the laſt obſervation. For he ſays *there was ſome remains of a parhelion at* I fig. 495. but it appeared very faint for want of matter which ſeldom extends ſo far. XL. Inverted arches below the corona's.

567. The way to find the ſhapes of thoſe inverted arches is this. Upon a ſpherical ſurface ABC with the pole B deſcribe a great circle ADC repreſenting the horizon; then let BED be a vertical circle paſſing through the ſun at E; ſo that DE may be the ſun's altitude at the time of obſervation. Now to find the inverted arch which touches the leſſer of the two corona's; in the vertical BED take EF and EG each equal to 22 degrees, and the points F, G will be the higheſt and loweſt points of the corona, and likewiſe the middle points of the inverted arches which we are going to determine. To find any other points of them by means of the horizontal cylinders in various poſitions, let us imagine a great many of them to have their baſes parallel to a vertical circle BM, and let the circle HEK be parallel to it and paſs through the ſun. Now if we recollect what has been demonſtrated above concerning the effects of upright cylinders[a], and if we imagine the circle BM to be our horizon in whoſe center N the ſpectator is placed; and if we ſuppoſe the circle HEK, parallel to BM, to be the white circle paſſing through the ſun; the arch HM will be the ſun's altitude above the horizon BM and alſo his altitude above the planes of the baſes of the cylinders we are now concerned with; for we ſuppoſed their baſes parallel to the circle BM. Things being thus conceived we have nothing elſe to do but to determine the points K, L in the circle HEK where the lateral parhelia would appear; which is eaſy to be done by the table in the xvii ſection. For example let the arch HM, which is imagined to meaſure the ſun's altitude above the baſes of the cylinders, be 30 degrees; and the table ſhews that each of the arches EK, EL are $28° 48'$; and one of the points K, L ſo determined, will be in the upper, and the other in the under inverted arch: and by the ſame means we may find as many points as are neceſſary to diſcover the ſhape and bending of the arches FK, GL. And by the ſame method we may determine the inverted arches that touch the higheſt and loweſt points of the larger corona. But before we diſmiſs the conſideration of this phænomenon, let us obſerve that the circle ZHY, fig. 495, might alſo be produced by theſe horizontal cylinders, although in the xxxvi ſection we aſcribed its origine to the cylinders in all manner of ſituations, or to the half thawed grains. For as we

LXI. How to find the ſhapes of the inverted arches. Fig. 497.

[a] Art. 543.

 have

have ſhewn in the xxi ſection how the upright cylinders produce a corona, ſo the horizontal ones may do the ſame thing; becauſe their extremities may be terminated by hemiſpheres or half ſpheroids. And it is very probable that this was the true cauſe of the circle *ZHY* in this phænomenon; becauſe we ſee it proceeds downwards on each ſide from the parhelion *H*; and we ſhall alſo obſerve the ſame thing in other parhelia. For this ſeems to intimate that the ſame horizontal cylinders produced both the circles we are ſpeaking of, and alſo the inverted arches.

LXII. *Scheiner's* obſervation in 1630 explained. Fig. 498.

568. Theſe things being underſtood we ſhall find no difficulty in explaining that other obſervation of *Scheiner*'s in 1630 in which there appeared ſix ſuns as deſcribed in the appendix. We need only obſerve that the parhelia *O* and *P* appeared in the interſections of the larger corona and of the white circle, becauſe this corona was produced by upright cylinders according to ſection xxi. Nevertheleſs ſince the higher parhelion *R* is alſo in this corona, the upper part of it may alſo be produced by horizontal cylinders in the manner abovementioned. There was no inverted arch obſerved to paſs through the parhelion *Q*, becauſe it might not be much extended on each ſide of *Q*; and therefore might not recede far enough from the corona *ZQβ* to appear diſtinct from it. For the ſun's altitude being $28\frac{1}{2}$ degrees, this arch was bent upwards but very little; becauſe we have already obſerved it was very flat when the ſun's height was 27 degrees. The cauſe of the double circle that compoſed the inner corona has been mentioned in the xxii ſection.

LXIII. A ſolution of 3 other phænomena obſerved by *Hevelius*. Fig. 499 to 501.

569. The cauſes of three other phænomena in the appendix, obſerved by *Hevelius* the 30 Mar. the 6 Apr. and the 17 Decemb. are ſufficiently evident from what has been ſaid. Except that in the laſt of them there appeared a white croſs, made by two beams of light, one vertical and the other horizontal, that croſſed each other in the diſk of the moon. The reaſon is evident from thoſe vertical and horizontal cylinders which produce the paraſelenæ and circles paſſing through them. For the horizontal beam was nothing but a piece of the large white circle that uſually paſſes through the ſun, as explained in the xiii ſection. The reaſon why this horizontal beam does not always appear may be the want of matter, that is of upright cylinders; or it may be extinguiſhed by the ſuperior brightneſs of the ſun or moon, as we obſerved in the xiv ſection.

XLIV. The vertical beam explained. Fig. 501.

570. The other beam that was perpendicular to the horizon was produced by reflection of the moon's rays from the cylinders that lay horizontal; by which alſo theſe inverted arches were made, as explained above. Now ſince the ſides and axes of theſe cylinders are parallel to the plane of the horizon though not to one another; it follows when the ſun's altitude is ſo ſmall as in this obſervation, (for it was but 12 degrees), that a great multitude of thoſe that appear to be ſpread from the horizon up to a certain diſtance above the moon, will reflect the moon's rays to the eye as if

if they came from this perpendicular beam: which will be the narroweſt and better defined at its edges next the moon than any where elſe; and the lower part between the moon and the horizon will be more perfect than the other part above the moon; where it ought to be dilated till it vaniſhes. Which ſeems to be agreeable to the phænomenon, ſo far as appears by the draught of it. In one thing indeed there is a difficulty in the theory; that is how an upright beam, cauſed by theſe reflections, could be ſo neatly terminated on each ſide as the figure ſeems to require; becauſe many of the horizontal cylinders that appear out of this beam may reflect rays to the eye; but the number of them is far leſs than the number of thoſe that lye in the beam. What has been ſaid of this bright beam may be demonſtrated by the 497 figure. For let *ADC* be the horizon, as before; *E* the moon whoſe altitude, meaſured by the vertical arch *ED*, is 12 degrees; *BM* another vertical ſuppoſed to be parallel to the baſes of a certain portion of the horizontal cylinders; *HEK* a leſſer circle parallel to *BM*; and of this portion of cylinders thoſe only that appear to lye in this circle *HEK* will reflect the moon's rays to the ſpectatot at *N*. In like manner every other portion of cylinders that have other particular directions, will have a lucid circle paſſing through the moon and extended down to the horizon. And as this circle *KEH*, in the parts next the moon at *E*, approaches very near to the vertical paſſing through the moon, ſo does a great number of other circles; ſo that by all of them taken together there is formed a bright ſort of a column paſſing through the moon's vertical *BED*. And this column will be ſo much the brighter and more conſpicuous as the moon is lower. Becauſe more of theſe lucid circles are crowded into a narrower ſpace, and are produced by a greater number of cylinders when the moon is low than when ſhe is higher elevated. Fig. 497.

571. This was alſo the cauſe of another meteor obſerved at *Caſſel* Jan. 2. 1586. by *Chriſtopher Rothman*; which he relates in this manner, in his deſcription of a comet in the year 1586. *The sky being very clear in the eaſt, juſt before ſun-riſe there appeared an upright column exactly ſituated in a vertical circle. Its breadth was every where equal to the ſun's diameter; and it looked as if ſome village was on fire beyond the mountains. For it appeared like a column of flame, excepting that its thickneſs was every where the ſame. Soon after in the ſame column there aroſe an image of the ſun exactly reſembling the true ſun. There was ſcarce one digit of this image under the horizon, when the true ſun began to riſe in the ſame column, which was followed in like manner by another image. The column with its three ſuns touching one another continued always upright or in a vertical circle, as appeared by the plummet of a quadrant. Theſe ſuns had all the ſame appearance except that the true ſun in the middle was brighter than the reſt. This appearance of the column paſſing through three ſuns laſted al-* XLV. Another meteor of the ſame ſort explained.

most a quarter of an hour, till they were covered by a black cloud descending from above.

The reflection of the sun's rays from the horizontal cylinders was the cause of this appearance; unless upon account of the images of the sun, it be more reasonable to suppose there was a large quantity of upright cylinders in the atmosphere, whose extremities were bounded by flat stars, such as *Des Cartes* has observed to fall down upon the ground[a]. For these cylinders swinging separate from one another (as I suppose) and being supported by a current of rising vapours, ought to continue suspended for the most part in that upright posture in which they were generated[b]; not exactly perhaps, but so as to decline but a little from it. These flat stars are therefore so many small planes, from which the rays of the sun are reflected in such a manner, that the superior part of the column with its parhelion, may be produced by reflection from the inferior bases of the cylinders; and the inferior part of the column with its parhelion from the superior bases. I observed that these cylinders might not be exactly upright but might incline a little, perhaps a degree or two; for if their bases were exactly plane and parallel to the horizon, they could not reflect any of the sun's rays to the eye.

[a] Fig. 504.

[b] Art. 538.

The rest is wanting in the original.

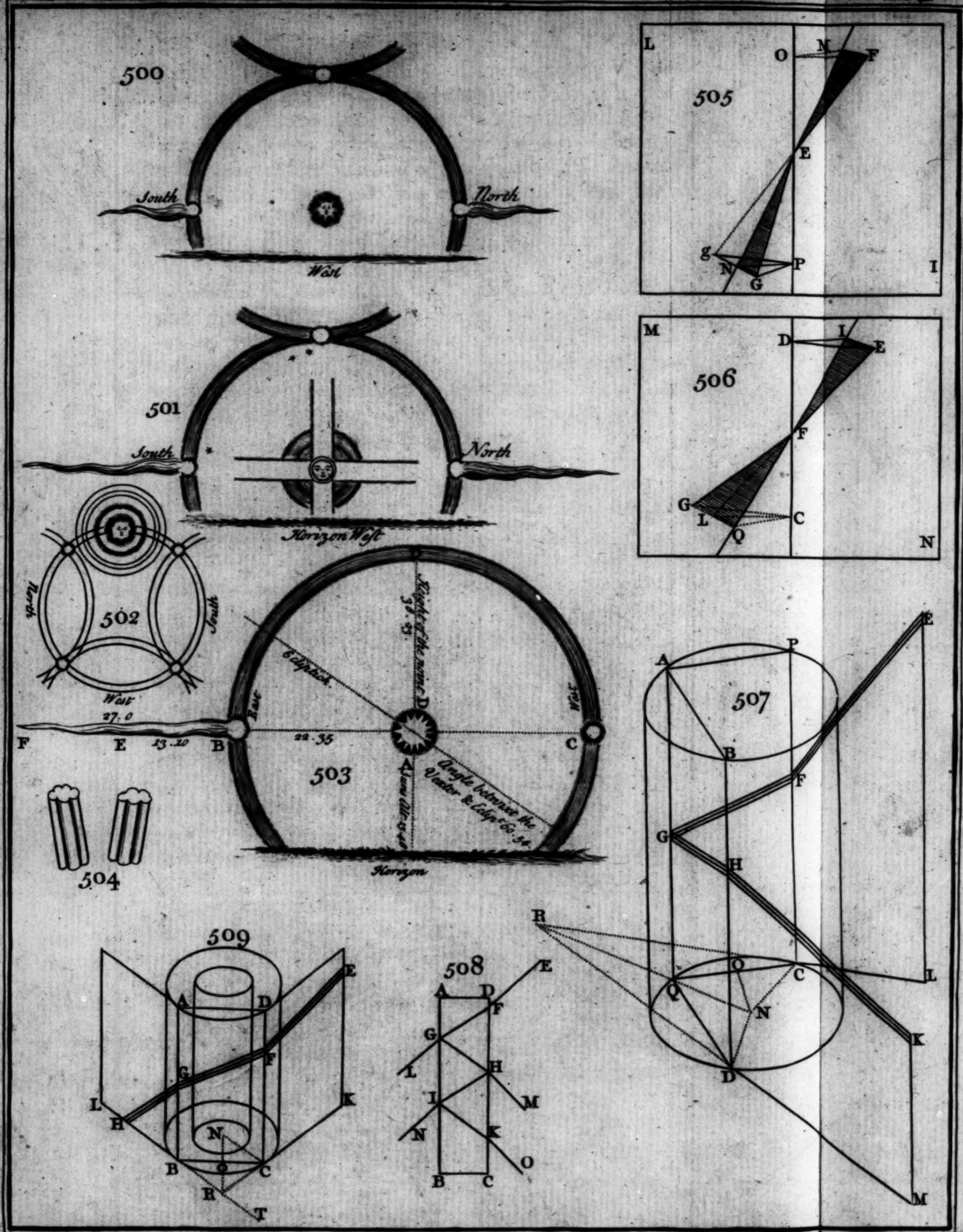
AN APPENDIX.

Containing the Observations quoted above, and a demonstration of the Tables.

I. Scheiner's *Observation of Corona's and Parhelia.* A. D. 1630.

Taken from *Gassendus*'s Works Tom. 6. p. 401. Fig. 498.

572. The diameter of the corona *MQN* next to the sun was about 45 degrees; and that of the remoter corona *ORP* was about 95°. 20′; they were coloured like the primary rain-bow, but the red was next the sun, and the other colours in the usual order. The breadths of all the arches were equal to one another, and about a third part less than the diameter of the sun, as represented in the scheme. Though I cannot say but the whitish circle *OGP* parallel to the horizon, was rather broader than the rest. The two parhelia *M, N* were lively enough; but the other two at *O* and *P* were not so brisk; *M* and *N* had a purple redness next the sun; and were white in the opposite parts; *O* and *P* were all over white. They all differed in their durations. For *P*, which shone but seldom and but faintly, vanished first of all, being covered by a collection of pretty thick clouds. The parhelion *O* continued constant for a great while though it was but faint. The two lateral parhelia *M, N* were seen constantly for three hours together: *M* was in a languishing state and dyed first, after seve-


500
South
North
West
501
South
North
Horizon West
502
North
South
West
27.0
503
Height of the moon
Ecliptick
East
West
F
E
13.20
B
22.35
C
Angle between the Vertex & Eclipt 60.54
Horizon
504
505
506
507
508
509

ſeveral ſtruggles; but *N* continued an hour after it at leaſt. Though I did not ſee the laſt end of it, yet I was ſure it was the only one that accompanied the true ſun for a long time, having eſcaped thoſe clouds and vapours which extinguiſhed the reſt. However it vaniſhed at laſt, upon the fall of ſome ſmall ſhowers. This phænomenon was obſerved to laſt 4 hours and a half at leaſt; and ſince it appeared in perfection when I firſt ſaw it, I am perſwaded its whole duration might be above five hours.

The parhelia *Q*, *R* were ſituated in a vertical plane paſſing through the eye at *F* and the ſun at *G*, in which vertical the arches *CRH*, *ORP* either croſſed or touched one another. Theſe parhelia were ſometimes brighter, ſometimes fainter than the reſt; but were not ſo perfect in their ſhape and whitiſh colour. They varied their magnitudes and colours according to the different temperature of the ſun's light at *G* and the matter that received it at *Q* and *R*; and therefore their light and colours were almoſt always fluctuating and continued as it were in a perpetual conflict. I took particular notice that they appeared almoſt the firſt and the laſt of all the parhelia excepting that at *N*.

The altitude of *Q* above the horizon, in the morning at the beginning of the obſervation, was $49^{\circ}\ 40'$; that of *R* was $76^{\circ}\ 10'$; that of the true ſun was $28^{\circ}\ 30'$. Hence the height of *Q* above the ſun was $21^{\circ}\ 10'$; and the height of *R* above the ſun was $47^{\circ}\ 40'$. There was a north wind at the beginning of theſe obſervations, but by degrees it changed to the eaſt and at laſt to the ſouth; yet it brought no very great nor laſting rains. For near a fortnight after the ſky looked always vapouriſh; and every day before dinner the ſun endeavoured to create new ſuns, but in vain, either for want of matter or of a due diſpoſition. For in the vertical circle I ſaw plainly ſome ſketches of parhelia for a long time. I ſaw alſo very manifeſt reciprocations of the lateral parhelia. The iris *ORP* ſeems to have been a portion of a ſingle circle concentrick to the ſun, but towards κ and θ it did not quite touch the horizon *AB*; and the lengths of the arches $O\kappa$, $P\theta$ were variable. The arches $ZQ\alpha$, $\beta Q\gamma$, $\delta\epsilon\zeta$, that immediately ſurrounded the ſun, ſeemed to the eye to compoſe a ſingle circumference, but it was confuſed, and had unequal breadths; nor did it conſtantly continue like it ſelf, but was perpetually fluctuating. But in reality it conſiſted of the arches expreſſed in the ſcheme, as I accurately obſerved for that very purpoſe. The horns *HRC* ſeemed to be a portion of a ſmaller circle touching the greater *ORP* in a contrary poſition in a common knot at *R*. The arches $ZQ\alpha$, $\beta Q\gamma$ cut each other in a knot at *Q* and there they formed a parhelion. The parhelia *N*, *M* ſprung out from the common interſections *M*, *N* of the Iris $\delta\epsilon\zeta$ and of the whitiſh circle *ONMP*. The north part of the ſky was clearer than the ſouth, which being overcaſt with ſlender vapours afforded more matter for this appearance.

II.

II. Hevelius's *Observation of Paraselenæ, seen at* Dantzick A.D. 1660. Mar. 30.

Fig. 499. 573. In the beginning at one a clock in the morning the moon *A* was surrounded by an entire whitish circle *BCDE*; in which there were two mock-moons at *B* and *D*, one at each side of the moon, consisting of various colours and shooting out very long and whitish beams by fits. That on the left hand extended its tail towards the thigh of Serpentarius, the other on the right extended its tail towards Jupiter, as represented in the figure. Afterwards, at two a clock, a larger circle surrounded the lesser and reached down to the horizon. The tops of both these circles were touched by coloured arches like inverted rain-bows. The inferior arch at *C* was a portion of a larger circle, and the superior a portion of a lesser. This extraordinary sight lasted near three hours; the outward great circle vanished first of all, then the larger inverted arch at *C*, and presently the lesser; and last of all the inner circle *BCDE* disappeared. The diameter of this inner circle and also of the superior arch was 45 degrees; that of the exterior circle and inferior arch was 90 degrees.

III. Hevelius's *Observation of Parhelia* A. D. 1660. Apr. 6.

Fig. 500. 574. At half an hour past 5 in the evening while the sun was descending towards the horizon, he was crowned with arches of circles of various colours like the rain-bow. In the corona on opposite sides of the sun there were two parhelia variously coloured, with pretty long and whitish tails pointing from the sun. Near the zenith, where the corona was a little faint and imperfect, there shone out another inverted arch, having a third parhelion in the middle of it, which appeared somewhat obscure. This phænomenon lasted half an hour, till sun-set, the sky being very clear. The inverted arch and the upper parhelion disappeared first; and then the parhelion on the left hand; but the third parhelion set with the true sun. The diameter of the corona round about the sun was about 45 degrees as I guessed by my eye.

IV. Hevelius's *Observation of Paraselenæ seen at* Dantzick A. D. 1660. Decemb. 17.

Fig. 501. 575. On the first day after the full moon at 30 minutes past 6 in the morning, the moon being 12 degrees high, I saw the moon in the west with 3 mock-moons about her, in this manner. The air being very clear at first I observed the moon surrounded with a double corona, [near her body as the figure seems to represent,] tinged with very bright and beautiful colours. On each side of the moon there was two arches of a large circle

cle about 45 degrees in diameter, which were also coloured like the rainbow, and exended down to the horizon. In which were two mock-moons with very long white tails. That on the left hand was near Procyon with a short tail; the other on the right hand had a longer tail. In the upper part where these collateral arches concurred, there was another arch inverted and variously coloured, with a third mock-moon in the middle of it, and somewhat duller than the other two. Moreover what was very extraordinary, there passed a large, white, rectangular cross through the disk of the moon; whose lower part reached down to the horizon; but on each side it did not quite touch the corona; as appears by the figure. It was so very bright and strong that it shone distinctly and clearly till sun-rise. But the mock-moons disappeared a little before.

V. *An Observation taken from* MATTHEW PARIS's *History*.

576. A wonderful sight was seen in England A.D. 1233. Apr. 8. in the 17th year of the reign of HENRY III. and lasted from sun-rise till noon. —— At the same time on the 8th of April about one a clock, in the borders of *Herefordshire* and *Worcestershire*, besides the true sun there appeared in the sky four mock-suns of a red colour; also a certain large circle of the colour of crystal, about two foot broad, which encompassed all England as it were. There went out semicircles from the side of it, in whose intersections the four mock-suns were situated; the true sun being in the east and the air very clear. And because this monstrous prodigy cannot be described by words I have represented it by a scheme, that shews immediately how the heavens were circled. The appe arancewas painted in this manner by many people, for the wonderful novelty of it. Fig. 502.

VI. *A true delineation of a parhelion seen at* Leyden A. D. 1653. Jan. 14. *between one and two a clock in the afternoon; and observed in the Academical Observatory by* SAMUEL CHAR. KECHELIUS *a* Hollenstein.

577. *BDC* was a white circle almost 35 min. broad; the altitude of its highest point *D* was 38° 23. Its center was in the sun, whose height was 15° 48, that is at 36 minutes past one; his Azimuth being 23° 40′ towards the west, and the angle made by his vertical circle and the ecliptick 60° 54′. The mock-suns *B*, *C* were oblong and unequal, at the distance of 22° 35′ on each side of the sun, and had the same altitudes as the sun. The western parhelion at *C* was the fainter of the two and changed from yellow to white and disappeared first; the eastern one at *B* was brighter, with a lucid arch shooting from the sun, and was coloured with purple, red and yellow; the shape of its tail *BF* was conical, 27 degrees long, the parhelion being the base of the cone; the part *BE*, 13° 10′ long, consisted of bright yellow and red light, the other part *EF* being whitish; which Fig. 503.

vanished

vanished before the parhelion did. It appeared for half an hour and lasted one quarter longer than *C*. And the corona disappeared a little after.

VII. HUGENIUS's *Observation* 13 May 1652.

578. I observed a circle about the sun in its center; its diameter was about 46 degrees; and its breadth the same as that of the common rainbow; it had also the same colours, though very weak and scarce discernable, but in a contrary order, the red being next the sun; and the blue very dilute and whitish. All the space within the circle was possessed by a duller vapour than the rest of the air; of such a texture as to obscure the sky with a sort of a continued cloud, but so thin that the blue sky-colour appeared through it. The wind blew very gently from the north.

A DEMONSTRATION *of the* CONSTRUCTION *of the* TABLES, *and of some other Mathematical Propositions relating to the foregoing Dissertation.*

PROPOSITION I.

579. *A ray of the sun reflected from any vertical plane makes the same angle with the horizon as the incident ray does; or, the sun seen by reflection from any vertical plane appears of the same height as when seen directly.*

Fig. 505. Let EF and EG be an incident and reflected ray from the vertical plane LI; and let a plane $MFEGN$ passing through the incident and reflected rays, cut the reflecting plane LI at right angles in the line MN. From any point G in the reflected ray draw GN perpendicular to MN and produce it through the plane LI till it meets the incident ray FE, produced through it in G. Lastly let $PGNg$ represent an horizontal plane cutting the vertical line OEP in P. Now since the angles MEF and NEG or NEg, which are the complements of the angles of incidence and reflection, are equal; and since the line GNg is perpendicular to and bisected by the horizontal line PN, the triangles NPG, NPg are also equal. Hence also the triangles PEG, PEg being both right angled at P, are also equal; and so the angles PgE, PGE, that is the altitudes of the incident and reflected rays are also equal. *Q. E. D.*

580. *Corol.* 1. Hence the vertical planes PEg, PEG passing through the incident and reflected rays, are equally inclined to the reflecting plane LI; their inclination to it being measured by the equal angles NPg, NPG.

581. *Corol.* 2. And if the sun be seen by rays successively reflected from the concave or convex sides of any number of cylinders, whose axes are vertical,

vertical, he will appear of the same altitude, as if seen by direct rays. For if the plane LI be conceived to touch the side of the cylinder along the vertical line OEP, the reflection at E from the side of the cylinder will be just the same as from the tangent plane[a]. a Art. 19.

582. *Corol.* 3. Hence also if the sun be seen by rays successively reflected from any number of vertical planes, he will appear of the same height as if seen by direct rays.

Proposition II.

583. *If a ray of the sun be refracted through a vertical plane in any position, the sine of the altitude of the incident ray above the horizon, will be to the sine of the altitude of the refracted ray above the horizon, in the given ratio of the sine of incidence to the sine of refraction.*

For let MN be the refracting vertical plane; EF an incident and FG the refracted ray; EF being supposed on the foreside and FG on the backside of the plane MN; and let a plane $EIFGQ$ passing through the rays EF, FG, cut the plane MN at right angles in the line IQ. This being the plane in which the refraction is made, take FE and FG of any equal lengths, and drawing EI and GQ perpendiculars to the plane MN, the angle IEF will be equal to the angle of incidence, and FGQ to the angle of refraction; whose sines are FI, FQ. Draw the vertical line DFC, and let two horizontal planes, passing through the horizontal lines IE, GQ, be represented by the triangles DEI, CGQ; and the angles FDI, FDE, and also FCQ, FCG will be right angles; and therefore the triangles FDI, FCQ will be similar. Therefore FI is to FQ as FD to FC. That is, the sine of incidence is to the sine of refraction, as the sine of the angle FED, the altitude of the incident ray, to the sine of FGC, the altitude of the refracted ray. For FE and FG were put equal, and FDE, FCG are vertical planes. *Q. E. D.* Fig. 506.

584. *Corol.* Hence the apparent altitude of the sun seen by refraction through any vertical plane, is always the same as it would be when the refracting plane is so situated that the plane of incidence is also a vertical plane; in which case the sines of the altitudes of the incident and of the refracted ray, are the very sines of incidence and refraction.

Proposition III.

585. *In any given altitude of the sun, the sines of the angles, which the vertical planes drawn through the incident and the refracted ray, do make with a vertical plane perpendicular to the refracting vertical plane, are in a given ratio; compounded of the ratio of the sines of the altitudes of the incident and of the refracted ray directly and of their cosines inversely.*

 Things

Fig. 506. Things remaining as they were, produce the incident ray EF along the plane of refraction FGQ till it cus GQ in L; and the line joining the points L, C is the horizontal section of the vertical plane FCL passing through the incident ray EF; and CG is the horizontal section of the vertical plane FCG passing through the refracted ray FG; and lastly GQ is the horizontal section of a third vertical plane conceived to stand perpendicular to the refracting plane MN. Therefore the inclinations of the two former planes to the latter, are measured by the angles CLQ, CGQ. Now in the triangle CLG, the sine of the angle CLG or of CLQ, is to the sine of CGQ, as CG, to CL*, or as $DF \times CG$ to $(DF \times CL =)$ $CF \times DE$; because DF is to CF as DE to CL. Now the ratio of $DF \times CG$ to $CF \times DE$, is compounded of the ratio of DF, the sine of the angle DEF which measures the altitude of the incident ray, to CF, the sine of the angle CGF which measures the altitude of the refracted ray; and of the ratio of CG, the cosine of the latter angle, to DE, the cosine of the former. Q. E. D.

* Art. 221.

586. *Corol.* Hence putting I to R for the ratio of the sines of incidence and refraction, the radius FE or $FG = r$, the sine of the sun's altitude $FD = s$; we have the cosine $DE = \sqrt{rr - ss}$, and the sine of the altitude of the refracted ray FG, that is $FC = \frac{R}{I} s$*, and its cosine $CG = \sqrt{rr - \frac{RR}{II} ss}$. Hence putting P to Q for the ratio of the sines of the angles CLQ, CGQ, by this proposition we have $\frac{P}{Q} = \frac{I}{R} \times \frac{\sqrt{rr - \frac{RR}{II} ss}}{\sqrt{rr - ss}}$; which will be of use hereafter. To compute this ratio, when s is the sine of 25 degrees and I to R as 250 to 187 (which is the ratio that *Hugenius* uses in the construction of his Tables) we have

* Art. 583.

Log. sin. 25° or the Logarithmick sine s - - - - - -	9.62594
Log. $\frac{I}{R}$ - - - - - - - - - - - - - - - - -	0.12610
Hence Log. $\frac{R}{I} s$ - - - - - - - - - - - - - -	9.49984
The Log. sine complement of this last sine, is the log. $\sqrt{rr - \frac{RR}{II} ss}$	9.97713
Log. sine compl. 25° is the log. $\sqrt{rr - ss}$ - - - - - -	9.95727
Hence Log. $\frac{\sqrt{rr - \frac{RR}{II} ss}}{\sqrt{rr - ss}}$ - - - - - - - - - - -	0.01986
Log. $\frac{I}{R}$ was - - - - - - - - - - - - - - - -	0.12610
Hence Log. $\frac{P}{Q} = \frac{I}{R} \times \frac{\sqrt{rr - \frac{RR}{II} ss}}{\sqrt{rr - ss}}$ is - - - - - - - -	0.14596

PROPOSITION IV.

587. *If a ray of the sun,* EF, *falling obliquely upon the outside of a pellucid upright cylinder,* AC, *whose bases are parallel to the horizon, be refracted into it at* F, *and be reflected within it at* G, *any number of times or no times at all, before it emerges out again by refraction at* H; *the sun seen by any emergent ray whatever, as* HK, *will appear of the same altitude as if he was seen by direct rays.* Fig. 507.

Through the points of incidence, reflection and refraction, *F, G, H,* draw the vertical lines *PC, AQ, BD*; and supposing vertical planes to touch the cylinder in every one of these lines, the refractions and reflections at these planes will be just the same as if they were made at the cylindrical surface. Put *I* to *R* for the ratio of refraction, and the ratio of the cosines of the angles *PFE, CFG,* (that is of the sines of the altitudes of the incident and refracted rays) is the same as *I* to *R**; and the ratio of the cosine of *CFG* or *AGF* or *QGH* or *GHB* (which are all equal[a]) is to the cosine of *KHD* as *R* to *I**; therefore by compounding these ratios the cosine of *PFE* is to the cosine of *KHD* in a ratio of equality, and consequently the angles themselves and their complements are equal. *Q. E. D.* * Art. 583. a Art. 581. * Art. 583.

ANOTHER DEMONSTRATION.

588. Let the parallelogram *ABCD* be a vertical section through the axis of the cylinder and through an incident ray *EF*, and since the refracted and reflected rays, *FG, GH, HI, IK,* &c. make equal angles with the parallel lines *AB, CD*; it is evident that the incident ray *EF* and the emergent ray *GL* or *HM* or *IN* or *KO,* &c. will be all inclined to the vertical lines in the same angles. And all other rays which fall obliquely on the cylinder, will also, after refractions and reflections, be inclined to their vertical lines in the same angles[b]. *Q. E. D.* Fig. 508. b Art. 581. 584.

PROPOSITION V.

589. *When the rays of the sun are so refracted through the sides of an erect cylinder, as to touch the side of a lesser cylinder upon the same axis, it is proposed to find the inclination of the vertical plane of incident rays, to the vertical plane of emergent rays, when the sun has any given altitude and the ratio of the diameters of the cylinders is given.*

Let three vertical planes passing through the incident, refracted, and emergent rays *EF, FG, GH,* be represented by the parallelograms *DCK, DCBA, ABL,* and let them cut the horizontal plane of the base of the cylinders in *CK, CB, CL.* From *N* the center of the base draw *NO* perpendicular to the chord *BC* and join *NC*; and having produced *CK, BL* till Fig. 509.

till they meet in *R*, the angle *KRL* or its supplement *KRT* will be the measure of the inclination of the planes *DCK*, *ABL*. And since *NC* is the horizontal section of a vertical plane *NCF* perpendicular to the refracting surface at *C* or *F*, or to a plane that touches the cylinder in the line *CF*; by art. 585 the sine of the angle *NCR* is to the sine of the angle *NCO*, in a ratio compounded of the sines of the altitudes of the incident and of the refracted rays directly and of their cosines inversely; and consequently is given by art. 586. But *NO* the sine of the angle *NCO* is given from the given proportion of *NC* to *NO*, that is of the semidiameters of the cylinders; therefore the angles *NCO*, *NCR* are both given, and also their difference *BCR* or *CBR*, which are equal; and the sum of these angles equals the external angle *KRT*. *Q. E. I.*

590. For example, to shew the construction of the Table in the xvii section[a]; when the sun's altitude was 25° we computed the Log. of *P* to *Q*, or of the given ratio of the sines of the angles *NCR*, *NCO*, and found it 0. 14596[b]. Now putting the ratio of the semidiameters of the cylinders, that is *NC* to *NO* as 1000 to 473, the angle *NCO* will be 28° 14′, whose Log. sine is 9. 67492; to which adding 0. 14596, the sum is 9. 82088, which is the Log. sine of 41° 27′ = ang. *NCR*. Hence the difference of these angles is 13° 13′ = ang. *OCR*, whose double 26° 26′ = ang. *KRT*, the distance of the lateral parhelion from the sun.

a Art. 553.

b Art. 586.

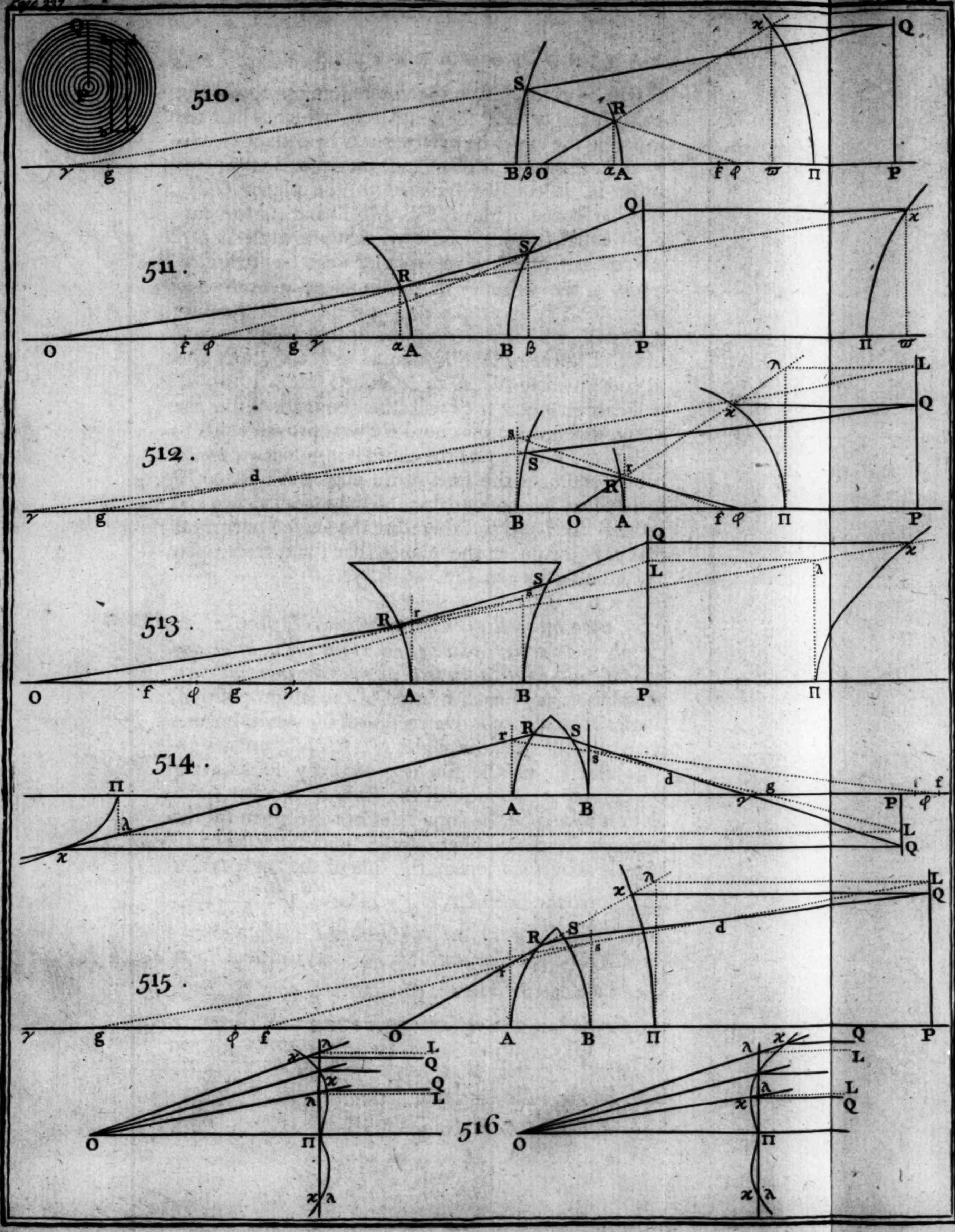
591. To find the greatest quantity of the angle *KRT*, let us suppose the plane *KCD* to touch the cylinder, or the angle *NCR* to be a right one, whose logarithmick sine is 10. 00000. Hence substracting 0. 14596 the log. of the ratio of the sines of *NCR*, *NCO*, there remains 9. 85404 for the logarithmick sine of the angle *NCO* = 45° 36′. Which being taken from *NCR* = 90° gives *RCO* = 44° 24′, whose double 88° 48′ = *KRT*, the greatest distance of the lateral parhelion from the sun when he is 25° high. In this case therefore *NO* : *NC* :: 714 : 1000.

PROPOSITION VI.

592. *When the rays of the sun are refracted through the side of an erect cylinder so as to touch the side of an inner concentrick cylinder of a given diameter, and are reflected at the surface of the outward cylinder any number of times before they emerge out again by refraction; it is proposed to find the inclination of the vertical plane of the incident rays to the vertical plane of the emergent rays.*

Fig. 507.

Case 1. Let *EF*, *FG*, *GH*, *HK*, be the course of the incident, refracted, reflected and emergent rays, and let as many vertical planes passing through them cut the horizontal plane of the base in *LC*, *CQ*, *QD*, *DM* respectively. Through *N* the center of the base produce *NQ*, and let it meet *LC* produced in *R*. Now from the given semidiameter *NO* of the inner cylinder, and the given ratio of the sines of the angles *NCR*, *NCO*, their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their difference OCR will be given, just as in the foregoing proposition; which being taken from NCQ or NQC gives NRC, which is half the angle LRM. For supposing the ray to be reflected at G by a plane touching the cylinder along GQ, this tangent plane and consequently the plane GQN perpendicular to it, is equally inclined to the planes GQC, GQD cor. 1. prop. 1. that is, the angles NQC, NQD and consequently NDQ, NCQ are all equal. Hence it follows, that the angle NDR is equal to NCR. For if the emergent ray be supposed to go backwards along KH, since by prop. 4. the altitudes of the incident and refracted rays at H are equal to those at F, by proposition 3 the ratio of the sines of the angles NDR, NDQ will also be the same as that of the sines of NCR, NCQ. Therefore as the angle NDQ is equal to NCQ, so NDR is equal to NCR and consequently NRD to NRC. Q. E. I.

593. *Case* 2. Now whatever number of reflections be made before the ray emerges again by refraction, since the chord CQ was proved equal to QD, for the same reasons QD will equal the chord lying under the next reflected ray, and so on; and since the sines of the angles NCR, NCQ are in a given ratio*, they may be considered as the sines of incidence and * Art. 585.
refraction of an horizontal ray LC; and therefore the angle under these incident and emergent rays, or under the planes that stand erect upon them, will be given in all cases by art. 488. *Q. E. I.*

Proposition VII.

594. *The sun shining upon an erect cylinder whose axis is vertical, to find the greatest angle which a vertical plane of emergent rays, after any given number of reflections, makes with a vertical plane of the incident rays.* Fig. 507.

Supposing the construction and demonstration of the last proposition, since the sines of the angles of incidence and refraction of the vertical planes of incident and refracted rays, that is of the angles NCR, NCQ are in a given ratio* when the altitude of the sun is given; by substituting * Art. 585.
the terms of this ratio instead of the ratio of the sines of incidence and refraction, this problem is solved by the same rules as were given for the rain-bow*. Thus putting m for the number of reflections increased by an * Art. 492.
unite, P to Q for the ratio abovementioned, the sine of the angle NCR will be $\frac{NC}{Q}\sqrt{\frac{mmQQ-PP}{mm-1}}$, and the sine of NCQ will be $\frac{NC}{P}\sqrt{\frac{mmQQ-PP}{mm-1}}$. For in Fig. 470 we had $BCq : BFq :: mmRR-RR : II-RR$, and disjointly $BCq : CFq :: mmRR-RR : mmRR-II$*. Whence $CFq = $ * Euc. I. 47.
$\frac{BCq}{RR}\times\frac{mmRR-II}{mm-1}$. Hence the angle required is given by art. 491. *Q. E. I.*

595. *Carol.* When there is but one reflection, $m=2$; and therefore putting $NC=r$, the sine of $NCR=\frac{r}{Q}\sqrt{\frac{4QQ-PP}{3}}=\frac{2}{\sqrt{3}}\sqrt{rr-\frac{PP}{4QQ}rr}$, which is an easier form for the following calculation.

596. To shew the construction of the Table in the xxviii section. For example, when the sun's height was 25° we found the Log. $\frac{P}{Q}$* 0. 14596

* Art. 586.

The Log. of r, or of the whole sine, is - - - -	10. 00000
Whence Log. of $\frac{P}{Q}r$ is - - - - - - - - - - -	10. 14596
Log. of the number 2 is - - - - - - - - - - - -	0. 30103
Hence Log. $\frac{P}{2Q}r$ is - - - - - - - - - - - -	9. 84493
The sine compl. of which log. or the log. of $\sqrt{rr-\frac{PP}{4QQ}rr}$ is	9. 85398
Log. $\frac{2}{\sqrt{3}}$ is - - - - - - - - - - - - - - - -	0. 06247
The Log. of $\frac{2}{\sqrt{3}}\sqrt{rr-\frac{PP}{4QQ}rr}$ is - - - - - - -	9. 91645
The angle NCR answering to this sine, is 55° 35′	
Log. $\frac{P}{Q}$ was - - - - - - - - - - - - - - -	0. 14596
Remains the log. of the sine of NCO - - - - - -	9. 77049

The angle NCO answering to that sine, is 36° 07′. Hence their difference $QCR = 19°\ 28'$, which being taken from NQC or $NCO = 36°\ 7'$, gives $QRC = 16°\ 39'$, whose double $LRM = 33°\ 18'$, the distance of either of the back parhelia from the vertical circle opposite to the sun, when his altitude is 25 degrees.

597. Hence it appears that if the semidiameter of the inward cylinder of snow, be greater than $\frac{590}{1000}$ of the semidiameter of the whole cylinder, it will hinder the appearance of the back parhelia, by stopping the refracted rays when the sun is 25° high; because in this case NO, the sine of the angle $NCO = 36°\ 07'$, is almost $\frac{590}{1000}$ of NC.

CHAPTER XII.

To determine the apparent shapes, positions, magnitudes and distances of large objects, seen by rays that fall upon reflecting or refracting surfaces, not only perpendicularly or almost perpendicularly, but with any degrees of obliquity.

PROPOSITION I.

598. WHEN *a large picture or a plane object of any given shape, stands perpendicularly to the common axis of any number of refracting or reflecting surfaces, its apparent shape, situation, magnitude and distance from the eye, at any point of the axis, may be found as follows.*

Let

Let $OABP$ be the axis of the reflecting or refracting surfaces AR BS; O the place of the eye; $ORSQ$ any reflected or refracted ray; QP a line perpendicular to the axis; $\varkappa$ the intersection of a line $Q\varkappa$, drawn parallel to the axis, and of the visual ray OR produced; $\Pi\varkappa$ a curve line described by the intersection $\varkappa$ while the visual angle AOR is diminished to nothing. Now while the whole figure is turned about the axis $OP\Pi$, the curve $\Pi\varkappa$ will describe a curve surface and PQ a circular plane. Let any part $abc\,d$ of this plane be the given object or picture; and while an indefinite line is carried round the out-lines of the picture $abcd$, and is kept parallel to the axis $P\Pi$, it will trace the out-lines of a corresponding picture upon the curve surface at $\Pi\varkappa$. I say the plane picture $abcd$ will appear by the reflected or refracted rays, in the same shape, situation, magnitude and distance, with which the curve picture at $\Pi\varkappa$ would appear to the naked eye at O; supposing the eye to be inverted and turned about when the curve $\Pi\varkappa$ falls behind it *, as in fig. 514. Fig. 510, 511. * Art. 139.

For imagine a physical circle described by the revolution of any point Q, about the axis OP, to be removed into the place of the equal circle described by the revolution of the corresponding point $\varkappa$; and if the surfaces AR, BS were taken away, its apparent shape, situation, magnitude and distance would be the same as before*. For since the visual rays as RO, $\varkappa O$ are always in one continued line, the shape, situation and magnitude of the picture upon the retina would be the same as before. And of consequence all the physical circles that compose the plane at PQ, when thus removed till they touch the curve surface at $\Pi\varkappa$, would severally and jointly appear to the naked eye at O, in the same shape, situation, magnitude and distance as they did before. What has been said of the whole circles, is also true of any given parts of them that compose any part $abcd$ of the circular plane at PQ; namely, that their several arches, which compose the plane figure $abcd$, when severally removed, by a motion parallel to the axis $P\Pi$, till they touch the curve surface $\Pi\varkappa$, will there compose a corresponding part of it, which would appear to the naked eye at O in the same shape, situation magnitude and distance, with which the plane $abcd$ appeared by the reflected or refracted rays. *Q. E. D.* * Art. 139.

599. *Corol.* 1. Hence if the plane surface at PQ and the curve surface at $\Pi\varkappa$, be both cut by any plane passing through the axis $P\Pi$ or parallel to it, the straight section of the former will appear in the same shape, situation, magnitude and distance, with which the curve section of the latter would appear to the naked eye remaining at O.

600. *Corol.* 2. Whenever the intersection $\varkappa$ describes a straight line perpendicular to the axis $O\Pi$, the object at PQ will then appear in its proper shape; that is, in the same shape, situation, magnitude and di-

stance

ſtance with which it would appear to the naked eye viewing it upon a perpendicular plane at the diſtance $O\Pi$.

LEMMA I.

601. *Let there be any number of indeterminate finite lines* x, y, z, *whoſe fluxions bear any finite ratio's to one another, and let another line* $v=\frac{xy}{z}$; *I ſay the fluxion of* v *will alſo bear a finite ratio to the fluxion of* x *or* y *or* z.

For by the known rule for finding fluxions we have $\dot{v}=\frac{\dot{x}y}{z}+\frac{\dot{y}x}{z}-\frac{\dot{z}xy}{zz}$ whoſe ſeveral parts bear finite ratio's to one another, as being compounded of none but finite ratio's; and conſequently the ratio of the whole value of $\dot{v}$ to any one part of it, is alſo finite; and therefore the ratio of $\dot{v}$ to $\dot{x}$ or $\dot{y}$ to $\dot{z}$ is alſo finite. *Q. E. D.*

LEMMA II.

Fig. 510, 511. 602. *If all the refracting or reflecting curves* AR, BS *cut their axis* $AB\Pi$ *at right angles, the curve* $\Pi\varkappa$ *will alſo cut it at right angles.*

For let $R\alpha$, $S\beta$, $\varkappa\varpi$ be perpendiculars to the axis, and let the rays RS, SQ be produced till they cut it in φ and γ. Then ſince the triangles $O\alpha R$, $O\varpi\varkappa$ are ſimilar, and alſo $\alpha\varphi R$, $\beta\varphi S$, and likewiſe $\beta\gamma S$, $P\gamma Q$; we have $O\alpha : O\varpi :: \alpha R : \varpi\varkappa$ or PQ, that is, in a ratio compounded of αR to βS, βS to PQ, that is, of $\alpha\varphi$ to $\beta\varphi$, $\beta\gamma$ to $P\gamma$. Hence $O\varpi = O\alpha \times \frac{\beta\varphi}{\alpha\varphi} \times \frac{P\gamma}{\beta\gamma}$. Now when the angle AOR is diminiſhed to nothing, the interſections φ, γ will move to the focus's f, g and coincide with them; and $O\varpi$ will become $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$. Therefore when the abſciſs $\Pi\varpi$ is vaniſhing, the vaniſhing increments or decrements of the finite lines $O\alpha$, $\alpha\varphi$, $\beta\varphi$, $\beta\gamma$, $P\gamma$ are theſe reſpectively, αA, $\alpha A \pm \varphi f$, $\beta B \pm \varphi f$, $\beta B \pm \gamma g$, γg. But the aberration φf is as αA*, and γg as βB*, and conſequently all theſe vaniſhing increments or decrements bear finite ratio's to one another. Therefore $\varpi\Pi$ the vaniſhing increment or decrement of $O\varpi$, bears a finite ratio to αA, by the firſt Lemma, becauſe the vaniſhing increments of quantities are as their fluxions. But the ultimate ratio of αA to αR is a ratio of minority infinitely great, becauſe the curve AR, is ſuppoſed to cut its axis at right angles, and the ratio of αR to $\varpi\varkappa$, being the ſame as $O\alpha$ to $O\varpi$, is finite. Conſequently the ultimate ratio of $\varpi\Pi$ to $\varpi\varkappa$, being compounded of $\varpi\Pi$ to αA, αA to αR, αR to $\varpi\varkappa$, is a ratio of minority infinitely great, and therefore the curve $\Pi\varkappa$ cuts the axis of the other curves at right angles, which is alſo its own axis, becauſe the curves AR, BS are ſuppoſed to be ſimilar and equal on each ſide of it.

* Art. 645.

603.

603. *Corol.* 1. Hence it follows that the curve $\Pi\varkappa$ cannot form an angle of contact with its axis at Π.

604. *Corol.* 2. The focus's f, g being given, the vertex Π of the curve $\Pi\varkappa$ is found by taking $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$.

605. *Corol.* 3. Though the curves AB, BS were such as could reflect or refract all the rays of a large pencil so accurately as to belong to the focus's f, g, yet by reason of the increments $\alpha A, \beta B$ the line $O\varpi$ would be variable, and consequently the intersection $\varkappa$ would still describe a curve.

606. *Corol.* 4. Though the refracting curves were changed into their tangents, yet the intersection $\varkappa$ would still describe a curve, by reason of the aberrations $f\varphi, g\gamma$ by these refractions*. * Art. 647.

607. *Corol.* 5. But if the reflecting curves were changed into their tangents, the intersection $\varkappa$ would describe a straight line perpendicular to the axis at Π; there being no aberrations by these reflections.

608. *Corol.* 6. Therefore an object at PQ appears exactly in its proper shape by reflections from plane surfaces*. * Art. 600.

PROPOSITION II.

609. *The reflecting or refracting curves, and their intervals from each other and from the eye and object, being given; it is proposed to find whether the object will appear convex or concave towards the eye.*

Things remaining as they were, let Ar, Bs be tangents to the curves AR, BS; and let one of the remotest rays OR cut the first tangent in r; join fr cutting the next tangent in s; join also gs cutting the object in L; and lastly draw the reflected or refracted rays RS, SQ, either by an accurate construction*, if needful, or only by observing which way their aberrations $f\varphi, g\gamma$ tend from the successive focus's f, g; which will always be evident by the shapes of causticks described in Art. 69 &c. or 452 and 478 &c. Fig. 512 to 515. * Art. 469.

610. *Cas.* 1. Then if it be evident from the position of the ray SQ and of the line sL, that PQ is longer than PL, and will continue so till the visual angle AOR is reduced to nothing; the object PQ will appear convex towards the eye: and on the contrary, if PQ be always less than PL, the object will appear concave; and in both cases more or less curved, *cæteris paribus*, as the ratio between PQ and PL is greater or less.

For let the lines $Q\varkappa$, $L\lambda$, drawn parallel to the axis, cut the ray OR, (produced) in $\varkappa$ and λ; and let $\lambda\Pi$ be perpendicular to the axis; and in like manner as the value of $O\varpi$ was found by similar triangles, in Lemma 2, we shall find $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$. Which being invariable, shews that while the angle $\Pi O\lambda$ is decreasing to nothing, the intersection λ describes a fixt perpendicular $\lambda\Pi$. The same point Π is also the vertex

 of

* Art. 604. of the curve described by the other intersection $\varkappa$*, and consequently $\Pi\lambda$ is a tangent to it by Lemma 2. Therefore if PQ be always longer than PL, and consequently $O\varkappa$ always longer than $O\lambda$, the curve $\Pi\varkappa$
* Art. 605. must be convex towards the eye*. And on the contrary, if PQ be always shorter than PL, and consequently $O\varkappa$ always shorter than $O\lambda$, the curve $\Pi\varkappa$ must be concave towards the eye.

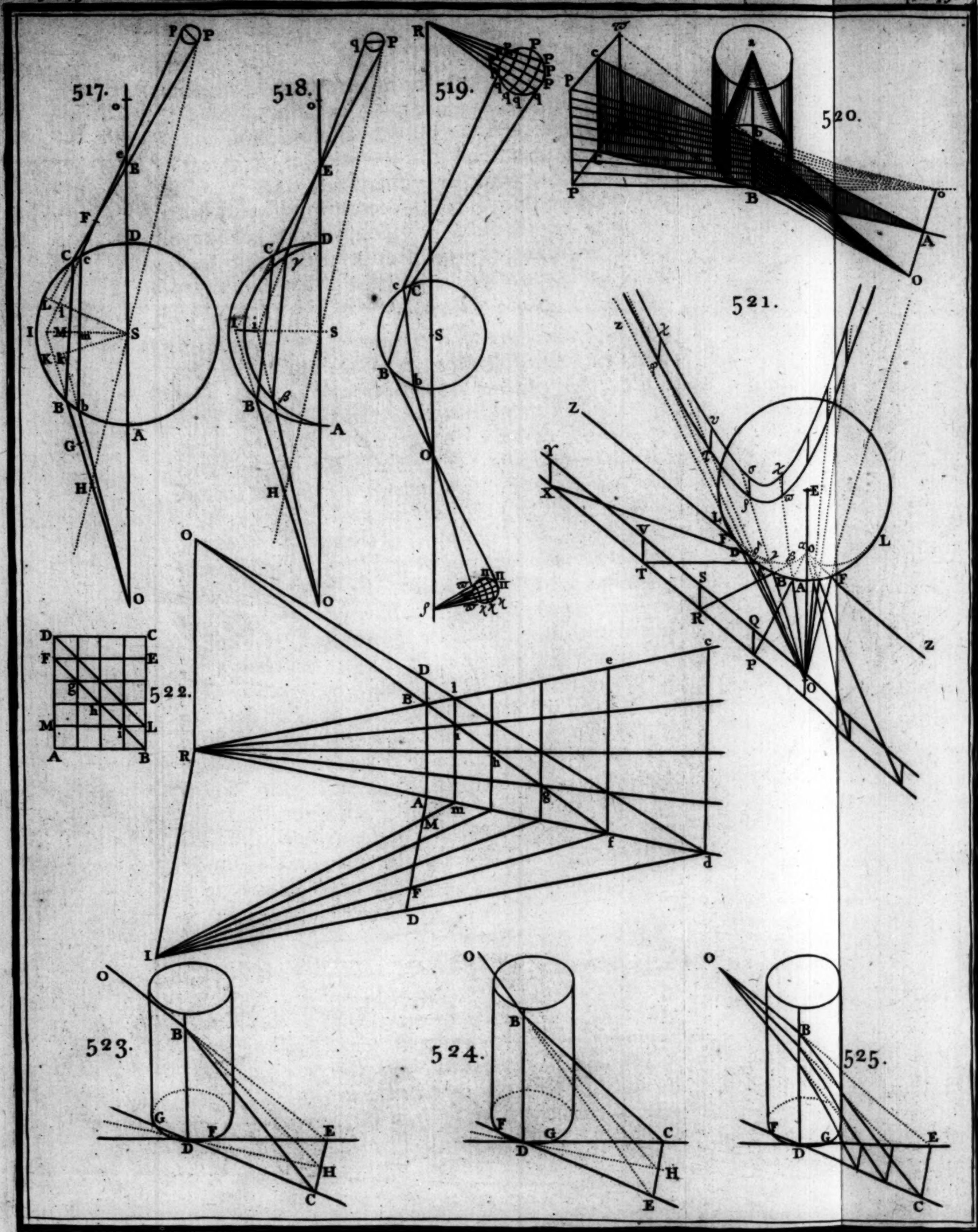
611. *Cas.* 2. Let SQ and sL (produced) intersect one another in d, and if the object be placed near their intersection, the curve described by d, while the angle AOR is diminishing, may cross the object; and then the curve described by $\varkappa$ will also cross its tangent $\Pi\lambda$ and form contrary flexures as represented separately in Fig. 516. For when d touches the object the intervals LQ, $\lambda\varkappa$ will be reduced to nothing, and will become negative, lying on the other sides of L and λ, after d has passed by the object. But by reason of the quick decrease of the angle LdQ or $gd\gamma$, the negative intervals LQ, $\lambda\varkappa$ will be become so small that the contrary flexure of the curve $\Pi\varkappa$ may scarce be perceptible by the eye. And if so, the object must appear convex or concave towards the eye according as the greatest PQ cut off by the remotest ray SQ, is longer or shorter than the greatest PL, as in the foregoing case.

612. *Cas.* 3. Consequently if the object touches the intersection d when remotest from the axis, it may be reckoned to appear straight in that situation or somewhere very near it: which is farther evident, because the object being carried for a considerable space from one side of d to the other, must change its apparent figure from concave to convex, or on the contrary by case 1.

Fig. 514. 613. *Cas.* 4. If the object is so placed that the points γ, g are on contrary sides of the point P, the points Q, L must also be on contrary sides of it. But by diminishing the angle AOR, the point γ in approaching towards g will pass by the point P, and then the points Q, L will be both on the same side of the axis; and so the rule in case 1 will soon take place. But the object will appear in a manner double, as might easily be explained, and so confused that its apparent shape will scarce be discernible even through a pin-hole.

Therefore the rule in the first of these cases is generally sufficient for determining the apparent convexity or concavity of an object, especially when it appears most evidently so. *Q. E. D.*

Fig 512 to 515. 614. *Corol.* Hence if we imagine a fictitious kind of rays as $OrsL$ to flow from the eye, and to be all reflected or refracted at the tangents Ar, Bs so accurately as to belong to the successive focus's f, g; the object PQ seen by such rays returning back in the same lines $LsrO$, would
* Art. 600. always appear accurately in its proper shape at the tangent $\Pi\lambda$*. The deformed appearances of objects are therefore caused both by the deflections of the curves AR, BS from their tangents and by the aberrations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tions of the rays. Upon both these accounts the minuter parts of large objects lying in order from the axis, become unequally or rather improperly magnified or diminished, and therefore appear at improper distances from the eye* and from each other; and thus the plane of the object is changed into a curve surface. For while the visual angle $AO\varkappa$ or AOR decreases uniformly, the equal lines PL, $\Pi\lambda$ decrease uniformly very nearly, or rather properly as they should do in vision with the naked eye, but PQ does not; therefore improper decrements of PQ appear under equal angles, and consequently are improperly magnified or diminished. * Art. 138. 139.

PROPOSITION III.

615. *Let the plane of this scheme represent a plane passing through the eye at* O, *and the center* S *of a refracting sphere* ABCD; *and let it be produced till it cuts the sun's disk in any line* Pp; *and out of an infinite number of rays supposed to flow in this plane from the eye at* O, *upon the great circle* ABCD, *let two of them, as* OBCFP, ObcFp, *fall upon the extremities of the chord* Pp *after refractions at the arches* Bb, Cc, *and after having crossed each other in the focus* F; *then from the center* S *let* SkK *be perpendiculars to the incident rays* Ob, OB *produced, and* SlL *perpendiculars to the emergent rays; and while the quantity of the refractions at* Bb, Cc, *is varied by any lateral motion of the sun, eye or sphere, in that plane of refraction, the apparent magnitude of the given chord* Pp *will vary directly as* FL *and inversely as* OK *very nearly, if the diameter of the sphere be very small in comparison to its distances from the eye and sun.* Fig. 517.

For then the angle LFl or PFp being but little more than half a degree at most, the angle KOk will also be but small, and consequently their perpendicular subtenses Kk, Ll will be very nearly equal. For drawing SMm perpendiculars to the rays BC, bc within the sphere; we have Mm to Kk and also to Ll, in the given ratio of the sines SM, SK or SL that determine the refractions*. Therefore Kk equals Ll and the angle KOk is to the angle LFl as FL to OK*. But since the chord Pp is given, the angle PFp or LFl is very nearly invariable, by reason of the immense distance and spherical figure of the sun; and consequently the apparent magnitude of Pp, measured by the angle KOk, is as FL directly and OK inversely very nearly. *Q. E. D.* * Art. 414. * Art. 222.

PROPOSITION IV.

616. *Things remaining as they were, let the ray* CP *cut the axis* OS *in* E, *and be produced till it meets the ray* OB *produced in* I; *and let the conical surface described by the ray* CEP, *turned about the axis* OS *into the position* γ Eq, *cut the disk of the sun in the arch* Pq; *and while the refractions of the visual rays* PCBO, q$\gamma\beta$O, *that lye in this conical surface, are varied by any motion of the sun, the eye or the sphere, the apparent magnitude of the given arch* Pq *will vary directly as* IE *and inversely as* IO *very nearly.* Fig. 518.

For let the point I describe the arch Ii by the revolution aforesaid; and since it is a common subtense of the angles IOi, IEi; the angle IOi is to IEi or PEq as IE to IO. But when the arch Pq is given, the angle PEq is very nearly invariable, by reason of the great distances of the sun and eye from the sphere, and consequently the apparent magnitude of Pq, measured by the angle IOi, is as IE directly and IO inversely very nearly. Q.E.D.

PROPOSITION V.

Fig. 517, 518. 617. *Things remaining as they were, let the refractions of the visual rays be varied by any motion of the sun, the eye or the sphere; and the apparent magnitude of the sun's disk will vary directly as the rectangle under* FL, EI, *and inversely as the rectangle under* OK, OI *very nearly.*

Fig. 519. For let innumerable planes of incidence and refraction cut the sun's disk in the lines Pp, Pp, &c. all converging to a point R in the axis OS produced; and let the same planes cut the picture of his disk upon the retina in as many corresponding lines $\Pi\varpi$, $\Pi\varpi$, &c. all converging to the point ρ in the said axis produced backwards. Again let innumerable conical surfaces, conceived to be described by the revolution of innumerable visual rays about the axis OS, cut the disk in the arches Pq, Pq, &c. and its picture upon the retina in as many corresponding arches $\Pi\kappa$, $\Pi\kappa$, &c. Then since every one of the lines $\Pi\varpi$, in the whole picture, is as the apparent magnitude of the corresponding lines Pp*, in the whole disk, that is as $\frac{FL}{OK}$*; the magnitude of the picture would also be as $\frac{FL}{OK}$, if the arches $\Pi\kappa$ were invariable. And in like manner since every one of the arches $\Pi\kappa$ in the same picture, is as the apparent magnitude of the corresponding arches Pq in the disk, that is as $\frac{IE}{IO}$*; the magnitude of the picture would also be as $\frac{IE}{IO}$, if the lines $\Pi\varpi$ were invariable. Therefore when the lines $\Pi\varpi$ and the arches $\Pi\kappa$ both vary, it is easy to understand that the magnitude of the sun's picture upon the retina, is as $\frac{FL}{OK} \times \frac{IE}{IO}$, that is, as the rectangle under FL, IE directly, and as the rectangle under OK, IO inversely very nearly. Q.E.D.

* Art. 91.
* Art. 615.
* Art. 616.

Fig. 517, 518. 618. *Corol.* 1. Let the refracting sphere be moved sideways from the rays that come directly from the sun to the eye, so as to describe a circle about the eye in any one plane of incidence and refraction; and if the diameter of the sphere be but small in comparison to its distances from the eye and the sun, the apparent magnitude of the given chord or diameter Pp will decrease perpetually, and very nearly as FL does, till it vanishes when the visual rays are tangents to the sphere. For when the visual ray OB touches the sphere, the line OK becomes equal to OB, and is then the least of all, and yet will differ but very little from its greatest magnitude OS. Therefore OK may be reckoned invariable, and consequently the apparent magnitude

tude of Pp is as FL*; which decreases perpetually and very fast till it be reduced to nothing[a]. * Art. 615. a Art. 439.

619. *Corol.* 2. Upon the same suppositions, the apparent magnitude of any given arch Pq upon the sun's disk, and consequently of the chord or diameter which joins its extremities, will decrease perpetually, and very nearly as IE does[b]. For when the angle which the refracting sphere subtends at the eye is but small, IE decreases perpetually during the motion above mentioned[c]. b Art. 616. c Art. 514.

620. *Corol.* 3. Hence upon the same suppositions as before, the apparent magnitude of the sun's disk will decrease perpetually and very nearly as the rectangle under FL, IE. And its apparent shape will grow more and more oval while the sphere is moved sideways more and more; its diameter Pp, in the plane of incidence and refraction, appearing shorter than the diameter perpendicular to it. Because FL decreases faster than IE; being always contained within it, except when they are both situated in the axis OS, and then they are equal to one another.

621. *Corol.* 4. Let o be the conjugate focus to O, and the area of the oval picture of the sun upon the retina, will be to its circular area when the sphere is exactly between the eye and the sun, as $FL \times IE$ to So^2. For when the sphere is exactly between the eye and sun, FL and IE become equal to each other and to So.

622. *Corol.* 5. In the ray $PCBGO$ let G be the focus of the rays that flow from P in any one plane of incidence and emergence; and H the focus of the rays that flow from P in the conical surfaces described by the same ray $PCBHO$ turned about the line PSH, the point H being the vertex of the cones; and the density of all the rays flowing from P, where they fall upon the cornea of the eye at O, will be as $GK \times HI$ directly, and as $GO \times OH$ inversely[d]; and consequently as $GK \times HI$, when the sphere is so remote as to subtend but a small angle at the eye. d Art. 521.

623. *Corol.* 6. Hence when a remote sphere subtends but a small angle at the eye, the sun's apparent brightness, seen through any part of it, is invariable. For the apparent brightness of any object is as the density of the rays in any one pencil directly, and as its apparent magnitude inversely; that is as $\frac{GK \times HI}{FL \times EI}$*. But since the rays that come from O and P, fall parallel or nearly parallel upon the circle $ABCD$, the lines GK and FL are nearly equal; and so are HI and IE. For the line SI bisects the angle EIH, and since SE is supposed to be nearly parallel to IH, and SH to IE, the angle EIS equals ISH, and HIS equals ISE; and consequently the triangles EIS, HIS are equicrural and equal to one another very nearly. * Art. 622. 620.

624. *Corol.* 7. Hence when the centers of innumerable refracting spheres are placed at equal intervals from one another in a large spherical surface whose center is the eye at O; this surface will appear illuminated by innumerable

numerable images of the ſun, all of the ſame brightneſs; but leſs and leſs in magnitude as they lye remoter from the ſun. And conſequently if the magnitude of the ſpheres be diminiſhed and their number be increaſed to infinity, the light of that ſpherical ſurface will appear uninterrupted, and ſtrongeſt next the ſun, and weaker perpetually in the parts that lye remoter from him; which agrees with our former demonſtration in the 525th article.

625. *Corol.* 8. What has been demonſtrated of the ſun is applicable to the moon, or to any round body, ſo ſmall or ſo remote from the ſphere as to ſubtend but a ſmall angle at any point of it; and is alſo agreeable to experience in viewing a lighted candle through a round bellyed jugg or decanter full of water, while it is moved ſideways from the rays that come directly to the eye.

626. *Corol.* 9. While the apparent magnitude of the candle decreaſes, its apparent diſtance will be found to increaſe, and on the contrary; and ſo its ſucceſſive apparent places will appear to deſcribe a curve; which in

a Art. 625. the caſe abovementioned [a] is convex towards the eye. And the apparent diſtance of the candle will vary reciprocally in a ſubduplicate ratio of the apparent magnitude of its ſurface, that is, reciprocally as a middle pro-

* Art. 620. portional between FL and IE*; or rather reciprocally as IE, that is

b Art. 616. 602. reciprocally as the apparent height of the candle [b]; becauſe its apparent breadth appears oblique to the viſual rays, by reaſon of the curvity of its ſucceſſive apparent places.

627. *Corol.* 10. For this reaſon any large plane ſurface, like a ſaſh-window, viewed through a ſphere will not appear plane but convex towards the eye. And beſides this convexity towards the eye, the top and bottom and ſides, and alſo the wooden bars that contain the glaſs, will appear concave towards the middle of the window, where a line drawn from the eye through the center of the ſphere would fall upon it. So that the whole appearance will be like the meridians upon a globe to the naked eye. For ſince the interior ſquares of glaſs are more magnified than the exterior ones, the intervals between the bars will be gradually diminiſhed in going outwards from the middle; which agrees with the manner of deſcribing theſe appearances by the laſt propoſition.

PROPOSITION VI.

Fig. 520. 628. *Any line* Pp *either ſtraight or crooked, which is ſeen from* O *by rays reflected from a ſtraight line* Bb, *drawn upon a plane ſurface* ABCcbA, *or upon the ſide of a cone or a cylinder, will appear in its proper ſhape; and the apparent diſtance of any point of it from the eye, will be equal to the whole courſe of the viſual ray.*

For let the reflecting plane $ABCcbA$ be generated by the motion of the line ABC about the axis OAo perpendicular to it. And let a ſuperficial

perficial pencil of rays OBb be ſuppoſed to flow from the eye at O, and after reflection from the ſtraight line Bb let them fall upon the ſurface of an object in the line Pp, either ſtraight or crooked. From every point of the object Pp draw the lines $P\Pi$, $p\varpi$ parallel to OA till they meet the viſual rays produced in Π, ϖ; and the apparent object $\Pi\varpi$ compoſed of all their interſections will be ſimilar and equal to Pp. For taking Ao equal to AO, all the reflected rays will flow from o*, and ſince all the triangles OBo, Obo, are equicrural, all the triangles $PB\Pi$, $Pb\varpi$, being ſimilar to the former, will alſo be equicrural. Therefore the compound line $OB + BP$ equals $OB\Pi$ equals oBP, and likewiſe $Ob\varpi$ equals obp; and conſequently the plane figure $O\Pi\varpi$, conſiſting of all the former lines, will be ſimilar and equal to the figure oPp conſiſting of all the latter. Therefore the Line $\Pi\varpi$ is ſimilar and equal to Pp; and conſequently is ſeen by reflection in the ſame place and ſhape as $\Pi\varpi$ would appear in, to the naked eye[a]. * Art. 202. a Art. 139.

The ſurface of a cone is generated by the revolution of one ſide of an angle about its other ſide, and the ſurface of a cylinder by the revolution of one ſide of a parallelogram about its oppoſite ſide. Conſequently ſuppoſing the reflecting line Bb to be laid upon either of theſe revolving lines, ſo as to coincide with it, the reflecting plane BAb may be conceived to touch the conical or cylindrical ſurface in the line Bb; and the reflections from Bb being the ſame whether they be made from the curve ſurfaces or from the tangent plane, it follows that the line Pp will appear ſtraight, or in its proper ſhape, by reflection from the cone or cylinder, as well as from the tangent plane. *Q. E. D.* Fig. 520.

629. *Corol.* 1. Hence one may determine the apparent magnitudes, diſtances and ſhapes of a ſeries of given objects ſeen by reflection from a cylinder, in this manner. Let the circle $ABCD$, whoſe center is E, be the baſe of an upright cylinder, or rather a circular ſection of it, parallel to its baſe and paſſing through the eye at O; and let this plane be extended and cut a row of trees, for example, in P, R, T, X, &c. Join OE and let $o\alpha\beta\gamma\delta$ be a cauſtick by reflection of innumerable rays ſuppoſed to flow from O; and let the lines $P\alpha$, $R\beta$, $T\gamma$, $X\delta$, &c. be ſo drawn as to touch the cauſtick in α, β, γ, δ, &c. and let them cut the reflecting circle in A, B, C, D, &c, join OA, OB, OC, OD, &c. and in each of them produced, take $A\varpi = AP$, $B\rho = BR$, $C\tau = CT$, $D\varphi = DX$, &c. Then ſuppoſing the trees PQ, RS, TV, XY, &c. to be removed and planted upright at $\varpi\kappa$, $\rho\sigma$, $\tau\upsilon$, $\varphi\chi$, &c. they would appear to the naked eye at O, if the cylinder was removed, of the ſame magnitudes, at the ſame diſtances and in the ſame ſhapes, as they appeared in by reflection from the cylinder when they ſtood in their proper places. And if regular curves be drawn through the points ϖ, ρ, τ, φ, &c. and through κ, σ, υ, χ, the crooked ſurface that lyes between theſe curves, will be the apparent place and ſhape of the ſurface $PQXY$. But the readieſt practical way is firſt to draw any ray OB and then its reflection BR, cutting PX in any point R, and to take $B\varrho = BR$ &c. Fig. 521.

630. *Corol.* 2. In like manner if another row of trees be parallel to the former, either on the same side of the cylinder or on the opposite one; the space that lyes between the representations of the bases of both rows, will be the apparent figure of the walk between the rows. Thus we have determined the representations of any planes either perpendicular or parallel to the base of the cylinder. Consequently the representation of any oblique plane, as of one side of a roof of a house or the like, may be found by determining the representations of the top and bottom of it.

The manner of drawing deformed pictures, which shall appear regular in a cylinder placed upon the plane of the draught, is partly the reverse of the foregoing method. In order to describe it more distinctly I premise the following lemma's.

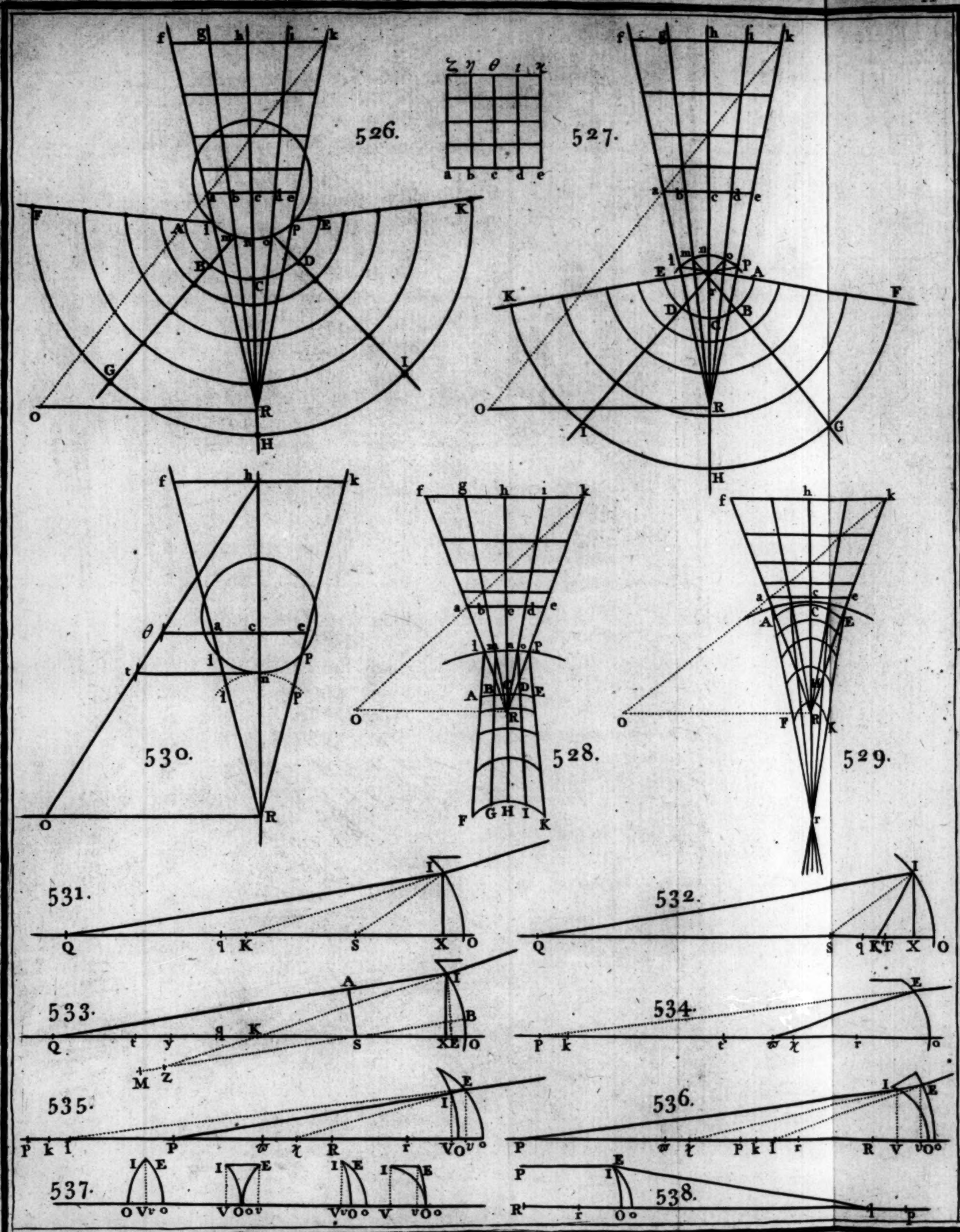
LEMMA I.

Fig. 522. 631. A rectangular parallelogram *ABCD* being divided into any number of little parallelograms, and being placed upright upon a plane *ABR*, (the line *AB* in one figure being placed upon *AB* in the other), it is proposed to describe its shadow *ABcd*, cast upon that plane by rays flowing from a given point *O*, placed at a given height *RO* above the plane of the shadow.

From the point *R* which lyes directly under the luminous point *O*, through the extremities and the divisions of the base *AB*, draw the lines *RAd*, *RBc*, &c. Then placing the line *AD*, equal to the height of the given parallelogram, and also *RI* equal to the height of the luminous point, upon the plane of the shadow, at right angles to *RA*; from *I* through the extremities and divisions of *AD*, draw *IDd*, *IFf*, *IMm*, &c. cutting the line *RA* produced in *d*, *f*, *m*, &c. And through these points *d*, *f*, *m*, &c. draw the lines *dc*, *fe*, *ml*, &c. parallel to *AB*; and the trapezium *ABcd* so divided into lesser trapeziums, will be the shadow of the parallelogram *ABCD* and of its divisions.

For conceiving the plane of the triangle *IRd* with all its lines, to be turned about the line *RAd*, till it stands at right angles to the plane *RAB*, the point *I* will then coincide with the luminous point in the air, and the divisions of the line *Ad* will be the shadows of the divisions of the side *AD* of the given parallelogram. And for the like reason the lines extended from *R* are the shadows of all the lines in the parallelogram which are perpendicular to *AB*; and the shadow *fe* of any line *FE*, parallel to *AB*, being the common intersection of two planes passing through the parallels *AB*, *FE*, is also parallel to *AB*. *Q. E. D.*

632. *Corol.* 1. If the given rectangle *ABCD* be contracted to a square *ABEF*, divided into any number of lesser squares, its shadow may be found more expeditiously; by drawing *RO* parallel to *AB* and equal to the height of *O*, and then by drawing *OBf*, cutting the shadows of the perpen-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perpendicular lines in the points $f, g, h, i,$ through which the parallel shadows must pass. For since AF equals AB, the point f is the same in both constructions; because Rf is to Af as RI to AF or as RO to AB. And since BF is a diagonal of all the little squares, its shadow Bf must be a diagonal of all their shadows.

633. *Corol.* 2. This construction is also applicable to a parallelogram, provided it be divided into little squares; by placing its side AD along the base AB, produced if need be, and by drawing OD cutting RA and RB produced in d and l. For dl is the shadow of the diagonal DL of the square $DCLM$, cut off from the parallelogram $ABCD$.

634. *Corol.* 3. Hence it follows that the shadow of the center of any square, may be found by drawing the diagonals of the corresponding trapezium.

635. *Corol.* 4. If the length of the shadow dA and the point R be given or assumed, the luminous point O may be found by drawing dD which will cut off the perpendicular RO required.

636. *Corol.* 5. Hence we may understand the method of drawing a copy of a picture, which shall appear deformed when viewed directly, and yet shall appear in just proportion when viewed from a certain point. For having described a parallelogram or a square about the original picture, and having divided it into any number of lesser squares, the more the better, and lastly having divided another square $ABEF$ of any size into the same number of lesser squares; first project its shadow $ABef$, and then transfer the parts and lineaments and colouring in every little square of the original picture, into the corresponding trapeziums of the shadow $ABef$. For to the eye of a spectator placed in the luminous point O, the deformed picture will appear reformed into the true proportions of the original; because the shape of the image of the whole picture and of all its parts upon the retina, is the same as if the rays had come to the eye originally from a regular picture drawn in miniature upon a square placed upright upon its base AB. And if the shades and colourings in the deformation be the same also as in the original picture, the spectator will judge it to be regular rather than deformed; by being more accustomed to view regular pictures in an upright position than deformed ones in an oblique position.

LEMMA II.

637. When a ray OB falls obliquely at B upon the convex or concave surface of an upright cylinder BFG, it is proposed to find the point C where the reflected ray BC will cut the plane of the base of the cylinder. Fig. 523, 524.

Draw BD perpendicular to the base cutting its circumference FG in D, and let DH be a tangent to it at D; then let the incident ray OB be produced till it cuts the plane of the base produced in E, and draw EH perpendicular to DH, and produce it till CH equals HE, and C will be

 the

the point where the reflected rays BC will fall upon the plane of the base produced.

For joining DC, DE and also BH, the plane of the triangle DBH will touch the cylindrical surface in the line BD; and since the perpendiculars HC, HE to this plane are equal, if the point C be supposed a focus of incident rays upon this plane produced, the point E will be their focus after reflection*. Therefore on the contrary, the ray OB tending towards E, will be reflected to C; and the reflection from the tangent plane is the same as from the cylindrick surface at B*. *Q. E. I.*

* Art. 202.

* Art. 19.

638. *Corol.* 1. A more commodious practice in order to the solution of the following problem, is to produce OB to E, and to draw DE cutting the circumference of the base in F, and to inscribe in it a chord DG equal to DF; and in DG produced to take DC equal to DE, which gives the point C where the reflected ray will fall upon the plane of the base. Because the equal chords DF, DG are equally inclined to the tangent DH.

Fig. 525.

639. *Corol.* 2. Hence several rays flowing from O upon the line BD, drawn along the side of the cylinder, will all be reflected to the straight line DC; where their intervals are equal to the respective intervals of the incident rays produced till they cut the line DE. Because each two distances DC, DE are equal.

640. *Corol.* 3. Several rays flowing from C upon BD, will all diverge from E; and therefore but one of them can fall upon the same point O.

641. *Corol.* 4. Any incident and reflected ray OB, BC produced, make equal angles with the line BD produced, and also with the plane of the base. For the triangles BDC, BDE are equal.

PROPOSITION VII.

Fig. 526 to 529.

642. *To paint upon a plane a deformed copy* ABCDEKIHGF, *of an original picture, which shall appear regular when seen from a given point* O, *by rays reflected from a polished cylinder, placed upon the circle* lnp, *equal to its given base.*

From the point R which lyes directly under O the place of the eye, draw two lines Ra, Re, which shall either touch the base of the cylinder, or else cut off two small equal segments from the sides of it; according as the copy is intended to be more or less deformed. Then taking the eye raised above R to the given height RO, somewhat greater than that of the cylinder, for a luminous point; describe the shadow $aekf$ of a square $ae\kappa\zeta$ (or parallelogram), standing upright upon its base ae any where behind the arch lnp, and divided after a similar manner to another square (or similar parallelogram) drawn upon the original picture[a]. Let the lines drawn from R to the extremities and divisions of the base a, b, c, d, e, cut the remotest parallel shadow in the points f, g, h, i, k, and the arch of the base in $l, m,$

a Art. 631.

n, o, p;

n, o, p; from which points draw the lines lAF, mBG, nCH, oDI, pEK, as if they were rays of light that came from a focus R, and were reflected from the base lnp; so that each couple as lA, lR produced may cut off equal segments from the circle. Lastly transfer the lines laf, mbg, &c. and all their parts in the same order, upon the respective lines lAF, mBG, &c. and having drawn regular curves, by estimation, through the points A, B, C, D, E, and through F, G, H, I, K, and through every intermediate order of points; the figure $ACEKHF$ so divided will be the deformed copy of the square drawn and divided upon the original picture; and will appear similar to it when seen in the polished cylinder, placed upon the base lnp, by the eye put into its given place O. And the same directions that were given in art. 637, serve for painting the deformed copy of the picture upon the deformed square AK.

For supposing the eye at O, raised to the height RO, to be a luminous point; every superficies of rays emitted from it towards every line of the square and consequently towards every line of its shadow ak, being intercepted by the surface of the cylinder, will be reflected to every corresponding line of the deformation AK; as will easily appear by comparing the solution of the lemmas and their corollaries with that of this problem. Therefore on the contrary, the rays which flow backwards from every line of the deformation AK, will be reflected to the eye as if they came directly from the lines of the shadow ak or from the lines of the square it self. Q. E. D.

643. *Corol.* 1. If the ratio of the sides of the original picture, and consequently of the similar parallelogram $ae\kappa\zeta$, be given; and nR be given or assumed; and nt, the height of the cylinder, or of the highest point of reflection, be also given; the height of the eye may be determined by placing nt perpendicular to nR, and by placing the parallelogram $ae\kappa\zeta$ upon its base ace at any convenient distance behind the arch lp, and by producing its base ca till $c\theta$ be equal to its height; then the line θt produced will cut off the required height RO; as will appear by conceiving the triangle $ROt\theta h$ to stand perpendicular to the paper upon the line $Rnch$. And if the data be varied, the rest may be easily determined by the relation of the lines we have been considering. Fig. 530.

644. *Corol.* 2. If the portion of the cylindrical surface, which reflects the light, be extended into a plane, the deformation AK will become exactly equal to the shadow ak. The 527, 528, and 529th figures belong to a concave cylindrical surface.

CHAPTER XIII.

The theory of the aberrations of rays is resumed and carried farther, in order to discover the limits of perfection of reflecting and refracting Microscopes; and to determine for them what was determined for Telescopes in the seventh chapter.

PROPOSITION I.

645. HAVING *the focus of homogeneal rays incident upon a spherical surface, to find the aberrations of the refracted or the reflected rays.*

Fig. 531,532. Let Q and q be conjugate focuses of incident and refracted or reflected rays [a]; QI an incident ray upon the spherical surface IO, whose semidiameter is SO; IK the refracted or reflected ray (produced) cutting the axis $OSqQ$ in K; IX the sine of the arch IO; and for the distances OQ, OS, OX putting Q, S, X; and m to n for the ratio of refraction, m being bigger than n; and θ for $m-n$; I say, if the ray be refracted at I, the aberration $qK = \overline{\frac{Q-S}{Q+\frac{n}{\theta}S}}\Big|^2 \times \frac{nnQ - nnS - mnS}{\theta m Q} X$; and if reflected, $qK = \overline{\frac{Q-S}{Q+\frac{1}{2}S}}\Big|^2 \times \frac{1}{2}X$.

a Art. 211.

Fig. 533. Draw $BSZM$ parallel to IQ, cutting IK produced in Z; and let SA be perpendicular to QI; and let t and M be the focuses of refracted rays, supposing the incident ones to have come parallel and the nearest to the semidiameters SO, SB; and by art. 224, $SM = St = \frac{n}{\theta}S$; and if the ray QI was parallel to SO, then would its aberration $ty = \frac{nn}{m\theta}X$ by art. 329; and by art. 333 the aberration $MZ : ty :: SA$ square $: IX$ square $:: QS$ square $: QI$ square or QO square*. Hence MZ or $Z = \frac{\overline{Q-S}|^2}{QQ} \times \frac{nn}{m\theta} X$; and $SZ = \frac{n}{\theta}S - Z$. Now because the triangles QKI, SKZ are similar; it is as $QK : SK :: QI : SZ$; and conjointly as $QK : QS :: QI : QI + SZ$. Whence putting I for QI, we have $QK = \frac{\overline{Q-S} \times I}{I + \frac{n}{\theta}S - Z}$, and therefore when the point I comes to O and K to q, the conjugate focus to Q, we have $Qq = \frac{\overline{Q-S} \times Q}{Q + \frac{n}{\theta}S}$; and so the aberration $qK = (QK - Qq =) \overline{Q-S} \times \overline{\frac{I}{I+\frac{n}{\theta}S-Z} -}$

* Art. 204.

Q

$\frac{\mathcal{Q}}{\mathcal{Q}+\frac{n}{\theta}S} = \overline{\mathcal{Q}-S} \times \frac{\overline{\mathcal{Q}Z-\frac{n}{\theta}S} \times \overline{\mathcal{Q}-I}}{\overline{I+\frac{n}{\theta}S-Z} \times \overline{\mathcal{Q}+\frac{n}{\theta}S}}$. With the ſemidiameter $\mathcal{Q}I$ deſcribe the arch IE cutting SO in E; and calling OE, E, we have $\mathcal{Q}-I=E$ and $I=\mathcal{Q}-E$; ſo then $qK = \frac{\overline{\mathcal{Q}-S} \times \overline{\mathcal{Q}Z-\frac{n}{\theta}SE}}{\overline{\mathcal{Q}+\frac{n}{\theta}S-Z-E} \times \overline{\mathcal{Q}+\frac{n}{\theta}S}} = \frac{\mathcal{Q}-S}{\overline{\mathcal{Q}+\frac{n}{\theta}S}\big|^2} \times \overline{\mathcal{Q}Z-\frac{n}{\theta}SE}$, becauſe $\overline{\mathcal{Q}+\frac{n}{\theta}S}\big|^2$ is vaſtly greater than $\overline{\mathcal{Q}+\frac{n}{\theta}S} \times \overline{Z+E}$. Now by art. 328, $XO : XE :: \mathcal{Q}E : SO$, and disjointly $XO : EO :: \mathcal{Q}E$ or $\mathcal{Q}O : \mathcal{Q}S$. Whence $E = \frac{\mathcal{Q}-S}{\mathcal{Q}} \times X$. For E and Z ſubſtitute their values, and the aberration $qK = \frac{\mathcal{Q}-S}{\mathcal{Q}+\frac{n}{\theta}S}\Big|^2 \times \frac{nn\mathcal{Q}-nnS-mnS}{\theta m\mathcal{Q}} X$.

Now if the ray be reflected at I, for n ſubſtitute $-m$ and conſequently $2m$ for θ, and the theorem changes into this, $qK = \frac{\mathcal{Q}-S}{\mathcal{Q}-\frac{1}{2}S}\Big|^2 \times \frac{1}{2}X$. For the calculation is the ſame whether the refracted ray goes backward or forward in the line IK; and to change the angle of refraction SIK into an angle of reflection, it (and its ſine n) muſt be diminiſhed to nothing, and then be made negative and equal to ($-m$ the ſine of) the angle of incidence $SI\mathcal{Q}$. And during this change the method of calculation continues unaltered. Q. E. J. Fig. 532.

646. *Corol.* 1. Put Y for IX, and for a refracted ray the aberration $qK = \frac{\mathcal{Q}-S}{\mathcal{Q}+\frac{n}{\theta}S}\Big|^2 \times \frac{nn\mathcal{Q}-nnS-mnS}{2\theta m\mathcal{Q}S} YY$, and for a reflected ray $qK = \frac{\mathcal{Q}-S}{\mathcal{Q}-\frac{1}{2}S}\Big|^2 \times \frac{1}{4}\frac{YY}{S}$. For $OX : XY :: XY : 2OS$ very nearly.

647. *Corol.* 2. Let S become infinite, and by the laſt corollary the aberration of a ray refracted at a plane ſurface, that is $qK = \frac{mm-nn}{mn} \times \frac{YY}{2\mathcal{Q}}$; and the aberration of a ray reflected at a plane ſurface is nothing at all.

648. *Corol.* 3. When the point of incidence is given, the longitudinal aberration qK of a reflected ray, from its focus q, is as Sq^2, the ſquare of the diſtance of that focus from the center of the ſurface. For by this propoſition qK is to $\frac{1}{2}X$, the aberration from T the focus of parallel rays[a], as $\mathcal{Q}S^2$ to $\mathcal{Q}T^2$, or as Sq^2 to ST^2; becauſe $\mathcal{Q}T$, ST, qT, being continual proportionals[b], are proportionable to their ſums or to their differences.

a Art. 334.
b Art. 207.

649.

649. *Corol.* 4. When Q and S are given the longitudinal aberration of the outmost reflected or refracted ray is as YY, the square of the semiaperture of the surface, by corol. 1.

Lemma I.

650. When the terms of a ratio involve two sorts of quantities, one of which is incomparably smaller than the other; the magnitude of that ratio is not altered by any alterations made in those infinitely small quantities.

This will be evident when applyed.

Lemma II.

Fig. 534. 651. In the produced semidiameter or of the spherical surface oE, let ϖ and p be conjugate focuses of refracted rays; also $\varkappa$ and k two other conjugate focuses; and when the interval $\varpi\varkappa$ of the focuses of incident rays is exceeding small, the interval pk of the focuses of the refracted rays will be $\frac{mnrr}{\overline{\theta\varpi - mr}|^2} \times \varpi\varkappa$; supposing m, n, θ to signify the same as in the former proposition.

For let t be the focus of parallel rays coming the contrary way to the incident ones that flow from ϖ, and by art. 224, $ot = \frac{m}{\theta}\, or$ and $rt = \frac{n}{\theta}\, or$; and by art. 238, $\varpi t : \varpi r :: \varpi o : \varpi p$, and conjointly $\varpi t : tr :: \varpi o : po$, that is (for or, $o\varpi$, ot, op putting r, ϖ, t, p) $\frac{m}{\theta} r - \varpi : \frac{n}{\theta} r :: \varpi : p = \frac{nr\varpi}{mr - \theta\varpi}$; and consequently putting k and $\varkappa$ instead of p and ϖ, we have $k = \frac{nr\varkappa}{mr - \theta\varkappa}$; and so the distance $pk = (p - k =)\ \frac{mnrr \times \overline{\varpi - \varkappa}}{\overline{mr - \theta\varpi} \times \overline{mr - \theta\varkappa}} = \frac{mnrr}{\overline{mr - \theta\varpi}|^2}$

* Art. 650. $\times \varpi\varkappa$* $= \frac{mnrr}{\overline{\theta\varpi - mr}|^2} \times \varpi\varkappa$, because the square of any quantity is the same whether its root be affirmative or negative. *Q. E. D.*

Proposition II.

652. ***Having the focus of homogeneal rays incident upon any lens, it is proposed to find the aberrations of the refracted rays.***

Fig. 535, 536. Let $OIEo$ be the given lens, whose vertexes are O and o; R the center of the first surface OI; r the center of the second oE; and in the axis $oOrR$

a Art. 211. let P be the focus of incident rays, and p the geometrical [a] focus of the refracted ones; it is proposed to find pl the aberration of the ray $PIEl$. From the points of incidence and emergence I, E, draw IV, Ev perpendicular to the axis, and putting D for the difference of the thicknesses Oo, Vv; P, R, r for the lines OP, OR, or; and 2 to 3 for the ratio of refraction in glass, I say the

$$\text{the aberration } pl = \frac{\left.\begin{matrix}-27RR\\+6Rr\\-7rr\end{matrix}\right\} PP \left.\begin{matrix}+66RRr\\+14Rrr\end{matrix}\right\} P - 52RRrr}{6PP\times\overline{R-r}|^2 - 24PRr\times\overline{R-r} + 24RRrr} D.$$

supposing the lines P, R, r to lye all on one side of the glass; and when they have different positions in different glasses, their signs in the theorem must be altered accordingly.

For let ϖ be the conjugate focus to P after the first refraction at I, and $I\kappa$ the first refracted ray; then calling VI and vE, I and E, and in cor. 1. prop. 1. for $\mathfrak{Q}$, S, m, n, θ, Υ put P, R, 3, 2, 1, I, and the first aberration $\varpi\kappa = \overline{\frac{P-R}{P+2R}}\Big|^2 \times \frac{4P-10R}{6PR}\times II$. Again taking κIE for the incident ray upon the second surface oE, let k be the conjugate focus to κ after the second refraction; then call $o\kappa$ and $O\varpi$, κ and ϖ; and in cor. 1. prop. 1. for $\mathfrak{Q}$, S, m, n, θ, Υ put κ, r, n, m, $-\theta$, E or I, (for they are equal, being in the ratio of κv to κV,) and the second aberration $kl = \overline{\frac{\kappa-r}{\kappa-\frac{m}{\theta}r}}\Bigg|^2 \times \frac{mmr+mnr-mm\kappa}{2\theta n\kappa r} \times II = \overline{\frac{\kappa-r}{\kappa-3r}}\Big|^2 \times \frac{15r-9\kappa}{4\kappa r} II = \overline{\frac{\varpi-r}{\varpi-3r}}\Big|^2 \times \frac{15r-9\varpi}{4\varpi r} II$*. But p is the conjugate focus to ϖ, and so by the 2d lemma, $pk = \frac{6rr}{\overline{\varpi-3r}|^2} \times (\varpi\kappa)\ \overline{\frac{P-R}{P+2R}}\Big|^2 \times \frac{4P-10R}{6PR} II$; and by art. 236, $\frac{3PR}{P+2R} = \varpi = \frac{3PR}{t}$, by putting t for $P+2R$. Substitute this value for ϖ; then $pk = \frac{8r^3\times\overline{P-R}^2\times\overline{2P-5R}}{\overline{RP-rt}|^3\times 36RrP} II$, and $kl = \frac{3\overline{RP-rt}|^2\times\overline{5rt-9RP}}{\overline{RP-rt}|^2\times 36RrP} II$. By adding pk and kl together and restoring the value of t and dividing all by 3, we have the total aberration

* Art. 650.

$$pl = \frac{\left.\begin{matrix}-27R^3\\+33RRr\\-13Rrr\\+7r^3\end{matrix}\right\} PP \left.\begin{matrix}+66R^3r\\-52R^2r^2\\-14Rr^3\end{matrix}\right\} P \begin{matrix}-52R^3rr\\+52R^2r^3\end{matrix}}{12RrPP\times\overline{R-r}|^2 - 48RRrrP\times\overline{R-r} + 48R^3r^3} II.$$

Now $OV = \frac{II}{2R}$ and $ov = \frac{II}{2r}$; whence $Vv = (OV + Oo - ov =)\ \frac{II}{2R} + Oo - \frac{II}{2r}$, and so $D = (Oo - Vv =)\ \frac{R-r}{2Rr} II$, and $II = \frac{2Rr}{R-r} D$. By putting this value instead of II and dividing every term by $R-r$, we have

pl

$$pl = \frac{\left.\begin{matrix}-27RR\\+6Rr\\-7rr\end{matrix}\right\} PP \left.\begin{matrix}+66RRr\\+14Rrr\end{matrix}\right\} P -52RRrr}{6PP \times \overline{R-r}|^2 - 24PRr \times \overline{R-r} + 24RRrr} D. \text{ Q.E.D.}$$

653. *Corol.* 1. When the incident ray is parallel to the axis, its aberration $pl = \frac{-27RR+6Rr-7rr}{12Rr \times \overline{R-r}} II$ or $= \frac{-27RR+6Rr-7rr}{6 \times \overline{R-r}|^2} D$. For when P is infinite the terms of the general theorems which contain PP are incomparably greater than all the rest.

Definition. Fig. 537. 654. Supposing a convex lens to have no thickness at its edges and a concave one to have none in the middle; when the ray falls upon the edge of either lens, then D becomes equal to the difference of the versed sines of the half arches that compose it. This I call the thickness of the lens, which being given we have the aberration of the extream ray by either of the theorems.

Fig. 538. 655. *Corol.* 2. The thickness D, focal distance p, and semidiameter R of the first surface, being given, the aberration of the extream ray coming parallel to the axis, that is $pl = \frac{27RR+24Rp+7pp}{6RR} D$, when the centers of the surfaces of the glass lye both on one side of it; but if they lye on contrary sides, $pl = \frac{27RR-24Rp+7pp}{6RR} D$. For by art. 234, the focal distance $p = \frac{2Rr}{R-r}$, whence $r = \frac{Rp}{p+2R}$, which being put for r in the theorem of cor. 1, gives the present one.

656. *Corol.* 3. The thickness D, focal distance p and semidiameter r of the second surface, being given, the aberration of the extream ray coming parallel to the axis, will be $pl = \frac{7pp+4rp+rr}{6rr} D$ when the centers of the surfaces lye both on the same side of the glass; but if they lye on contrary sides, it will be $\frac{7pp-4rp+rr}{6rr} D$; found by putting $\frac{rp}{p-2r}$ for R in cor. 1.

657. *Corol.* 4. When the focus of incident rays upon a given lens is given, or when they are parallel, the aberrations of the rays from the conjugate focus are as, II, the squares of the distances of the points of incidence or emergence from the axis of the lens. For in this case P, R, r being given, pl is as II.

LEMMA II.

658. The focal distance, half the breadth, and the thickness of any lens, are continual proportionals.

At

At the latter end of the demonstration of prop. 2. we had $II = \frac{2Rr}{R-r} D$, and by art. 232 the focal distance $p = \frac{2Rr}{R-r}$; whence $II = pD$, or p, I, D are continual proportionals. *Q. E. D.*

659. *Corol.* 1. Hence in glasses of all sorts of shapes, (that is whatever be the ratio and position of the semidiameters of their surfaces,) if any two of these three things, focal distance, breadth and thickness, be the same, the third is the same also.

PROPOSITION III.

660. *To compare the aberrations caused by the sphericalness of the figure of all sorts of glasses: and to determine the semidiameters of a glass which shall make the least aberrations.*

To make a just comparison we must suppose all our glasses to have the same focal distance, the same breadth, and consequently the same thickness by art. 659; and to differ only in their shapes, arising from the various magnitudes and positions of the semidiameters of their surfaces.

661. First then, I say, that when parallel rays fall upon the plane side of a plano-convex glass, the aberration of the extream ray, which is $\frac{9}{2}$ of the thickness, is less than the like aberration caused by any meniscus glass whose concave side is exposed to the incident ray.

662. Secondly, I say that when the said glasses have their convexities turned to the incident rays, the aberration of the extream ray in the plano-convex, which is now but $\frac{7}{6}$ of its thickness, is less than the like aberration of any meniscus in this position.

663. Thirdly, I say that a double convex glass, when the semidiameter of the first surface, upon which the rays fall, is to that of the second from whence they emerge as 2 to 5, is just as good as the plano-convex in its best position; the aberrations of both being $\frac{7}{6}$ of their common thickness.

664. Fourthly, I say that when the semidiameters of a double convex are equal, it is not so good as a plano-convex in its best position, its aberration being $\frac{5}{3}$ of its thickness: but if the semidiameters of its first and second surfaces be as 1 to 6, it is the best glass of all; the aberration of the extream ray being now but $\frac{15}{14}$ths of its thickness; which is the least possible; there being no such thing in nature as a glass composed of two spherical surfaces that has no aberrations. But if this best glass be inverted it becomes much worse; for the aberration will then be $\frac{145}{42}$ of its thickness.

665. Lastly, I say that when a plano-concave has its plane side exposed to parallel rays, the aberration of the extream ray is also $\frac{9}{2}$ of its thickness, and that when it is inverted, the aberration becomes only $\frac{7}{6}$ths; which is

less

leſs than the aberration of any concavo-convex glaſs; and equal to that of a double concave glaſs the ſemidiameter of whoſe firſt ſurface is to that of the ſecond as 2 to 5; and that the beſt of all double concave glaſſes has the ſemidiameters of its firſt and ſecond concavities as 1 to 6; and conſequently this is the beſt figure of a glaſs to help ſhort-ſighted perſons, as the double convex one of the like figure is the beſt for ſpectacles.

DEMONSTRATION.

666. Firſt then, by article 655. the aberration cauſed by a meniſcus is $\frac{2}{2}D + \frac{4p}{R}D + \frac{7}{6}\frac{pp}{RR}D$; which decreaſes continually as R increaſes, till when R is infinite and the meniſcus becomes a plano-convex, the aberration becomes equal to $\frac{2}{2}D$, the other terms being incomparably leſs than this.

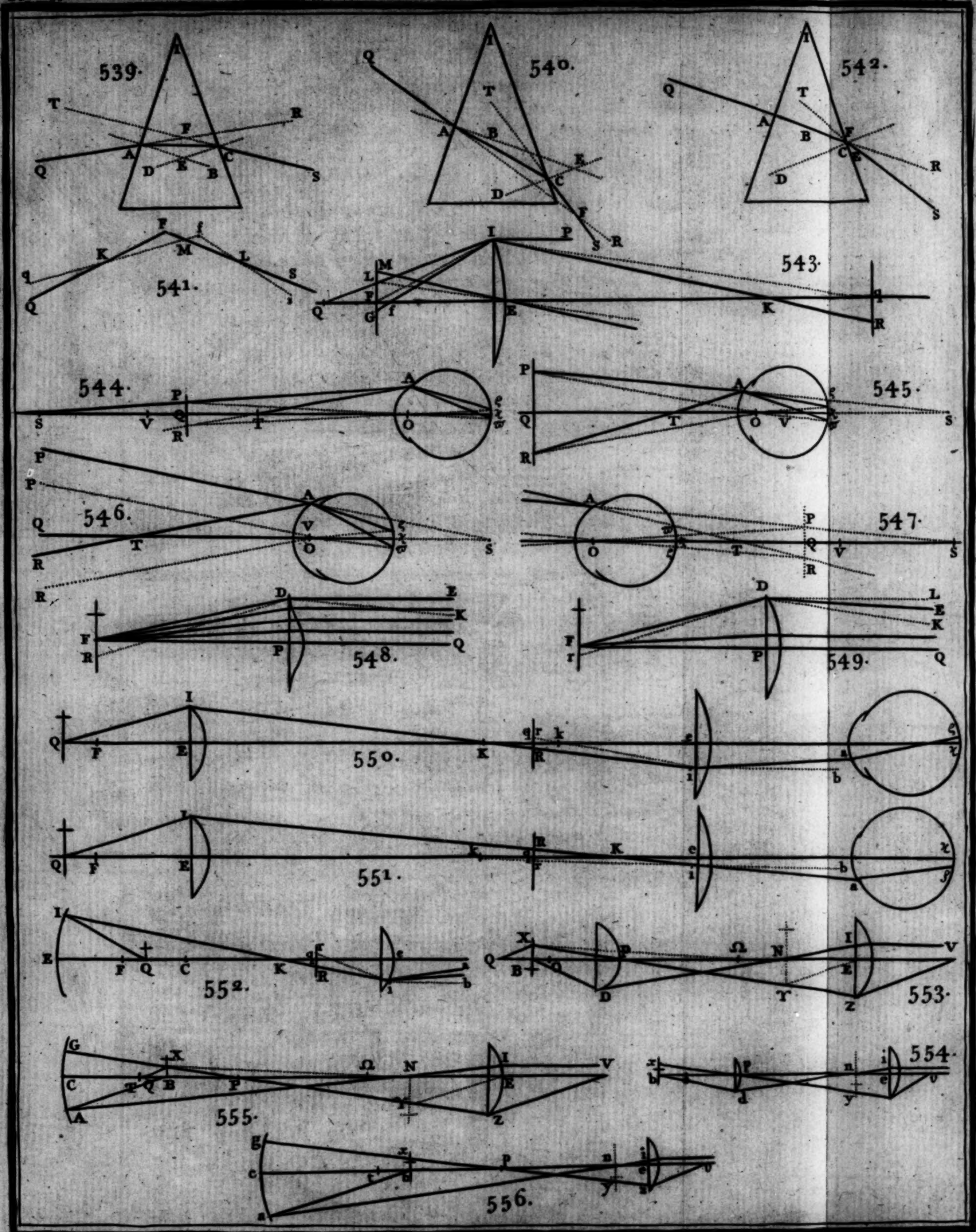
667. Secondly, after R is infinite let it become negative; and after that let r alſo become infinite and then negative, though bigger than R, that R may belong to the firſt ſurface whoſe convexity is now expoſed to the incident rays; and by art. 656. the aberration, now cauſed by the meniſcus, namely $\frac{7}{6}\frac{pp}{rr}D + \frac{2p}{3r}D + \frac{7}{6}D$* decreaſes continually as r the ſemidiameter of its ſecond ſurface increaſes, till it equals $\frac{7}{6}D$, when the glaſs becomes a plano-convex: which is leſs than its aberration $\frac{2}{2}D$, in the contrary poſition, in the ratio of 7 to 27, or almoſt 4 times.

668. Thirdly, to change this glaſs into a double convex, after r is infinite let it become negative, and ſuppoſing it vaſtly bigger than p, the negative term $\frac{2p}{r}D$, in the preſent aberration $\frac{7}{6}\frac{pp}{rr}D - \frac{2p}{3r}D + \frac{7}{6}D$, will be bigger than the affirmative one $\frac{7}{6}\frac{pp}{rr}D$; (the root of any proper fraction being bigger than its ſquare in a duplicate ratio of the terms;) conſequently their negative difference will make the aberration of a double convex leſs than $\frac{7}{6}D$, the aberration of a plano-convex; ſo long as that difference continues negative; but ſuppoſing r of ſuch a magnitude as will make that difference nothing, or $\frac{7}{6}\frac{pp}{rr}D - \frac{2p}{3r}D = 0$, the aberration of this double convex will be $\frac{7}{6}D$, the ſame as that of the plano-convex. Now for p put its value $\frac{2Rr}{R-r}$ and you have $\frac{7}{3} \times \frac{R}{R-r} - \frac{2}{3} = 0$, whence $5R = 2r$, that is $R : r :: 2 : 5$.

669. Fourthly, the aberration of any double convex glaſs whoſe ſemidiameters are given may be found by computing this quantity $\frac{27RR + 6Rr + 7rr}{6 \times \overline{R+r}|^2}D = pl$ by art. 653, the ſign of r or R being changed, be-

* See the table of Errata for p. 256.

cauſe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cause they now lye on contrary sides of the glass. Thus supposing $R=2$, $r=5$, the aberration comes out $\frac{7}{6}D$, as was proved just now; and supposing $R=r$, it comes out $\frac{4}{3}D$. To find the ratio of R to r when the aberration is the least possible, put $R=1$, then the aberration is $\frac{27+6r+7rr}{6\times\overline{1+2r+rr}}\times D=pl$. Now when any quantity or fraction comes to its least (or greatest) magnitude, it varies that magnitude but very little, or rather not at all while r varies but little. Suppose r increased by an exceeding small increment n, and by substituting $r+n$ for r, $rr+2rn+nn$ for rr, we have $\frac{6}{D}pl=\frac{27+6r+7rr}{1+2r+rr}=\frac{27+6r+7rr+6n+14rn+7nn}{1+2r+rr+2n+2rn+nn}$. Which gives this proportion, $27+6r+7rr:1+2r+rr::27+6r+7rr+6n+14rn+7nn:1+2r+rr+2n+2rn+nn$ (and disjointly) $::6n+14rn+7nn:2n+2rn+nn::3+7r:1+r$, neglecting the squares of n as inconsiderable[a]; and alternately $27+6r$ a Art. 650.
$+7rr:3+7r::(1+2r+rr:1+r::)\ 1+r:1$. Hence $27+6r+7rr=3+10r+7rr$, and by reduction $r=6$. But R was put 1, and so $R:r::1:6$. Put these values for R and r, and you will find the aberration $pl=\frac{15}{14}D$. Now if this best glass be inverted, that is, if we put $R=6$ and $r=1$, you will find $pl=\frac{145}{42}D$.

670. Since this glass in its best position has the least possible aberration, it is impossible for any spherical glass to have no aberration at all; which may also be proved after this manner. Supposing it possible that pl may be nothing, then putting $r=1$, we have $27\pm6r+7rr=0$ whence by reduction $r=\pm\frac{3}{7}\pm\sqrt{\frac{9}{49}-\frac{27}{7}}$, which is an impossible quantity; and so the ratio of R to r is impossible upon supposition that the aberration is nothing.

671. Lastly it appears by art. 655 and 656, that those theorems serve equally for concave and for convex glasses, and so the same demonstrations serve also for this last article. *Q. E. D.*

LEMMA IV.

672. If the angles of incidence and refraction of a ray, $QACS$, that Fig. 539, 540.
passes through a very small angle of a prism, AIC, be so little as to be reckoned proportionable to their sines; the angle of deviation RFS, contained under the incident ray $QAFR$ and the emergent ray $SCFT$ produced, will be to the refracting angle AIC, as the difference of the sines of incidence and refraction to the lesser of them; and consequently the magnitude of the angle of deviation RFS will be invariable in all positions of the ray.

For let the perpendicular AB, to the first surface AI, cross CD, the perpendicular to the second, in E; and supposing the ray AC to go both

ways out of the prism, the angle of incidence ACD will be to the angle of emergence DCT, in the given ratio of the sine of incidence to the sine of refraction, that is of i to r; and disjointly, we have ACD to ACT as i to $r-i$; and the angle CAB is to CAR, in the same ratio, supposing the ray to go backward along CA; and conjointly or disjointly we have $ACD \pm CAB$ to $ACT \pm CAR$, that is AED or AIC to RFS in the same given ratio of i to $r-i$. Q. E. D.

673. *Corol.* 1. Hence any two homogeneal emergent rays produced, will be inclined to one another in the same angle as the two incident rays are inclined to one another. For let the two incident rays QF, qf (produced) meet in K; and let the emergent rays SF, sf (produced meet in L; and let one of the incident rays cross the other emergent ray in M; and since, in the triangles KMF, LMf, the angles at M are equal and also those at F and f by this lemma, it follows that the remaining angles at K and L are also equal.

Fig. 541.

674. *Corol.* 2. And when two homogeneal rays are refracted through the same point of any lens, whose thickness is inconsiderable, the angle contained under their incident parts is equal to the angle under their emergent parts. For the thickness of the lens being very small, if the rays have a common point of incidence, their points of emergence will be very near one another; or if the point of emergence be common to both rays, their points of incidence will be very near one another; and still nearer if the rays cross one another within the lens; consequently the refractions through the lens will be nearly the same as through two planes, that touch its surfaces at two given points, near the points of incidence and emergence and contain a given angle with one another.

Fig. 72, 73.

675. *Corol.* 3. When the ray AC within the prism, coincides with a perpendicular to either of the planes, as with AB; one of the refractions will vanish at A; and then the angle of deviation RFS made by the other single refraction, will continue the same in quantity as before, when it was made by two refractions; because the magnitude of the angle of deviation is invariable by this lemma.

Fig. 542.

676. *Corol.* 4. Therefore when an heterogeneal ray is separated into coloured rays, by small refractions through a small refracting angle of a given quantity, the emergent rays of given colours will be inclined to one another and to the incident ray in certain given angles, in all positions of the incident ray. Because these inclinations made by two refractions, are every where equal to the inclinations made by a single refraction at the second plane, when the incident ray falls perpendicular upon the first plane.

677. *Corol.* 5. And an heterogeneal ray, refracted through a given point in a lens, has the same property as in the prism; that is the emergent rays

of

of given colours are inclined to one another and to the incident ray in given angles in all positions; for the reason mentioned in cor. 2.

678. *Corol.* 6. Therefore if several glasses of several sorts or shapes have the same focal distance, and the same aperture; the diameter of the circle of aberrations of heterogeneal parallel rays from their principal focus, will be the same in them all; being the same as in a plano-convex glass when its plane side is turned to the incident rays [a]; and in this case we have determined it above in art. 324. And when the rays in the incident pencil are either parallel or inclined to the axis of the lens, the diameter of the circle of aberrations is as its distance from the lens; because the angle RAS is invariable. a Art. 675. 676. Fig. 376.

679. *Corol.* 7. Therefore with respect to these aberrations by colours separately considered, it is indifferent which side of a lens is turned to the incident rays; because its focal distance is the same in both positions [b]. b Art. 233.

PROPOSITION IV.

680. *When* Q *the focus of homogeneal incident rays is not much farther from a lens* EI *than its focal distance* EF; *(as in double microscopes;) the lateral aberration* qR *of the outermost refracted ray* IR, *from their geometrical focus* q, *is to* FG *the lateral aberration (from the principal focus) of a ray* PI *that comes the contrary way, parallel to the axis and through the same point* I, *as* Eq *to* EF *nearly; that is directly as the distances of these focuses of refracted rays from the lens.*

Produce the perpendicular GF till it cuts the incident ray QI in L; and joining LE and Iq they will be parallel. For the triangles QLF, QIE being similar, we have $QL : QI :: (QF : QE ::) QE : Qq$*; which shews that the triangles QLE, QIq are equiangular. Draw EM parallel to the refracted ray IKR cutting FL produced in M; then the angle GIL equals PIK* or IKE or MEF; and consequently GL equals FM nearly; being the subtenses of the equal angles GIL, MEF and being very small and nearly perpendicular to their legs when QF and EI are very small. Hence taking away the common line FL, we have LM equal to the aberration FG. But the angles qIR, LEM, under two couple of parallel lines, are equal; and their subtenses qR, LM being equally inclined to their legs, we have $qR : LM$ or $FG :: (qI : EL ::) qE : EF$; because the triangles qIE, ELF are equiangular. *Q. E. D.* Fig. 543. * Art. 239. * Art. 674.

681. *Corol.* 1. When a plano-convex glass, which is nearly as good as the best of all [c], has its plane side turned towards the focus Q, we have the lateral aberration caused by its figure, that is $qR = \frac{7}{6} \times \frac{EI^3}{EF^3} \times Eq$, and the longitudinal aberration $qK = \frac{7}{6} \times \frac{EI^2 \times Eq^2}{EF^3}$: and in spherical lenses of any c Art. 662. 664.

shape

ſhape or ſubſtance, their aberrations are as theſe quantities. For the longitudinal aberration of the parallel ray PI, that is Ff is $\frac{7}{6}$ of the thickneſs of the glaſs [a] or $\frac{7}{6} \times \frac{EI^2}{EF}$ *; and becauſe the triangles fFG, fEI are equiangular, we have the lateral aberration $FG = \frac{7}{6} \times \frac{EI^3}{EF^2}$; and therefore by this propoſition we have $qR = \frac{7}{6} \times \frac{EI^3}{EF^3} Eq$; and becauſe the triangles qRK, EIK are equiangular, we have $qK = \frac{7}{6} \times \frac{EI^2 \times Eq^2}{EF^3}$.

a Art. 662.
* Art. 658.

682. *Corol.* 2. Hence FQ being given, the lateral aberrations, cauſed by the figure of any given lens, are as the cubes, and the longitudinal aberrations as the ſquares, of the diameters of the apertures.

683. *Corol.* 3. And hence the diameter of a circle of theſe aberrations cauſed by the figure only, that is $\frac{1}{2} qR^* = \frac{7}{12} \times \frac{EI^3}{EF^3} \times Eq$. Becauſe the demonſtration of the 339th article has this foundation only, that the longitudinal aberrations are as the ſquares, and the lateral ones as the cubes, of the linear apertures; and conſequently ſerves for any lens by cor. 2. as well as for a plano-convex one.

* Art. 339.

684. *Corol.* 4. The diameter of a circle of aberrations that will juſt contain all ſorts of heterogeneal rays flowing from Q, is equal to $\frac{2}{55} \times \frac{Eq}{EF} EI$, by art. 324 and 678, taking F and q for the focuſes of the mean refrangible rays.

685. *Corol.* 5. Therefore when the plane ſide of the object-glaſs EI, of a double microſcope, is turned towards the object at Q, the diameter of the circle of aberrations in its image at q, that would ariſe in homogeneal rays from the ſpherical figure only, would be to the diameter of a circle of aberrations of heterogeneal rays, if the figure of the lens was perfect, as 1 to $\frac{24}{385} \times \frac{EF^2}{EI^2}$. And conſequently theſe circles will be equal when the diameter of the aperture of the object-glaſs is half its focal diſtance very nearly.

LEMMA V.

686. Let innumerable pencils of rays be ſuppoſed to belong to innumerable focuſes in any given part of the axis of the eye produced both ways; and let it be propoſed to determine the leaſt circle upon the retina, into which all the rays, that enter the pupil, can be collected.

Fig. 544 to 547.

Let AO be the ſemidiameter of the pupil, and OTS the axis of the eye produced; and let pencils of rays belong to every point of the part ST, when its extremities S, T are both on the ſame ſide of the eye; or elſe to every point of the infinite axis, excepting the part ST, when S and T are on contrary ſides of the eye. Biſect ST in V, and towards the eye take

VQ

VQ to VT as VT to VO; and through Q draw a line PQR perpendicular to the axis, cutting SA and TA, the outermost rays of the extream pencils, in P and R respectively. And when the eye adapts it self to view the line PR as distinctly as possibly, its picture $\varpi\rho$ upon the retina will be the diameter of the least circle into which all the rays can be collected.

For since we made $VO : VT$ or $VS :: VS : VQ$, conjointly we have $VO + VS$ to $VS + VQ$ in the same ratio; and also disjointly $VO - VT$ to $VT - VQ$ in the same ratio; that is $SO : SQ :: TO : TQ$. Consequently, since the triangles SAO, SPQ and also TAO, TRQ are equiangular, we have $AO : PQ :: (SO : SQ :: TO : TQ ::) AO : QR$, and therefore the lines QP, QR are equal. Let $\varpi\kappa\rho$ be a distinct picture of PQR, that is let all the rays that may be supposed to flow from P, be collected to ϖ upon the retina; and the ray SPA being one of them, will go to ϖ along with the rest; and in like manner the ray RTA will go to ρ; and the lines PQ, QR being equal, their pictures $\varpi\kappa$, $\kappa\varrho$ will also be equal. Imagine the whole figure to be turned about the axis STO; and it will appear that the rays of one extream pencil, whose focus is S, will be scattered all over a circle upon the retina, whose center is κ and semidiameter is $\varpi\kappa$; and likewise the rays of the other extream pencil, whose focus is T, will be scattered all over a concentrick circle, whose semidiameter is $\kappa\varrho$; and these two circles will exactly coincide when their semidiameters $\kappa\varpi$, $\kappa\varrho$ are equal. But if the object PQR be supposed to approach toward the eye, the semidiameters PQ and $\varpi\kappa$ will both increase; and on the other hand, if PQR be supposed to recede from the eye, the semidiameters QR and $\kappa\varrho$ will both increase, while the other semidiameters PQ and $\varpi\kappa$ are decreasing; and in both suppositions the circle upon the retina which contains the rays of both the extream pencils, being the larger of the two, will be larger than it was before, when it was equal to the other; that is when the eye viewed the equal lines PQ, QR distinctly.

Now when S and T are on the same side of the eye, all the pencils that belong to intermediate focuses between the extreams S, T will fall within the circle above defined. For conceiving the nearest point T to recede from the eye, the lines QR and $\kappa\varrho$ will both decrease, and so will QP and $\kappa\varpi$ while S approaches towards the eye. And if S and T be on contrary sides of the eye, all the pencils will be collected into the said circle, whose focuses are in every point of the infinite axis excepting the part ST. For by conceiving both OS and OT to increase, QP and QR, and consequently $\kappa\varpi$ and $\kappa\varrho$ will both decrease. *Q. E. D.*

687. *Corol.* 1. Hence when a pencil of rays that flow from a single point of an object, shall be so disposed by reflections and refractions, as in falling upon the eye, to belong to innumerable different points of its axis

axis produced, as in the proposition; the diameter $\varpi\varrho$ of the least circle upon the retina, that contains them all, will be as the angle SAT; that is as the greatest angle in which the two outermost rays intersect one another at the pupil of the eye. For $\varpi\varrho$ is as the angle POR or PAR or SAT.

Fig. 546. 688. *Corol.* 2. Consequently when OS and OT are on contrary sides of the eye, and are equal to one another; that is when the two outermost rays are equally inclined to the axis and contrary ways, the diameter $\varpi\varrho$ of the least circle of aberrations, will be as the angle AST or ATS in which either of the outermost rays is inclined to the axis: and in this case the eye must be adapted to collect the parallel rays to a distinct point upon the retina. For the line VO being nothing in this case, the third proportional VQ becomes infinite; and $\varpi\varrho$ being always as the angle POR or PAR is now as its half, AST or ATS.

Definition.

689. In vision either with the naked eye or with glasses, the apparent indistinctness of a given object, is as the area of the least circle upon the retina, into which all the rays of a single pencil can be collected by the eye.

The reason of this definition has been shewn in the beginning of the demonstration of the 343d article.

Proposition V.

690. *In microscopes made with single lenses, a given object placed at their principal focuses will appear equally distinct, if their linear apertures be as their focal distances.*

Fig. 548. a Art. 81. 691. *Case* 1. At first let us suppose the figure of the lens DP, to be such as would cause no aberrations of homogeneal rays[a]; then if the object be placed at F, and FP be the focal distance of mean refrangible rays, all these rays that flow from F, will be refracted into the lines DE parallel to the axis FPQ. Let the violet contained in the outermost heterogeneal ray FD be refracted along DK; and let another violet ray be supposed to come backwards along ED, and to be refracted along DR, meeting the object in R. Then the apparent indistinctness of the point F being as the area of the least circle of aberrations, upon the retina, of the rays in one pencil flowing from F*, will be given when the diameter of this circle is given; or when the angle EDK* or FDR* is given; that is when the ratio of FR to FD or FP is given. But in heterogeneal rays the ratio of FR to DP is also given[b]; and consequently the apparent indistinctness will be given when the ratio of DP to PF is given.

* Art. 689. * Art. 688. * Art. 674. b Art. 324. 678.

692.

692. *Case* 2. Now let the figure of the lens DP become spherical while it keeps the same focal distance; and the innermost of the mean refrangible rays flowing from F, will still be refracted into lines parallel to the axis; but the outermost of them as DE will now converge towards it by too great a refraction [a]. Imagine a mean refrangible ray to come backwards along LD parallel to the axis, and to be refracted along Dr, so that Fr may be the lateral aberration of homogeneal rays caused by the figure of the lens. And the apparent indistinctness of the object so far as it depends upon this sort of aberration, will be given, as in case 1, when the ratio of Fr to FP, that is, of $\frac{DP^3}{FP^2}$* to FP, and consequently of DP^3 to FP^3, and of DP to PF, is given, as before in case 1. Fig. 549. [a] Art. 81. * Art. 681.

693. *Case* 3. Now let the violet contained in the heterogeneal ray FD be refracted along DK; and the angle EDK between the green and violet will be a little bigger than in the first case; being increased by a 27th or 28th part of the angle EDL*, which measures the increase of refraction or deviation caused by the change of the figure of the lens. This increment of EDK is therefore as the angle EDL, and consequently is given when the ratio of DP to PF is given, by case 1. But the apparent indistinctness will be given, when the diameter of the least circle of aberrations of all the rays upon the retina is given; that is when the whole angle KDL is given [b]; or when all its parts EDL, EDK and the increment of EDK, are given; and we have shewn that all these parts are given when the ratio of DP to PF is given. *Q. E. D.* Fig. 549. * Art. 324. [b] Art. 688.

694. *Case* 4. In these cases I have considered the aberrations of the violet rays from the green or mean refrangible ones, which were supposed to be parallel to the axis; and the red rays being inclined to the axis very nearly as much as the violet, and in a contrary position, will all be contained in the same circle of aberrations upon the retina [c]. *Q. E. D.* [c] Art. 688.

695. In microscopical lenses whose focal distances are not much longer than half an inch, there is no need to contract their apertures, for procuring distinct vision; the pupil it self being small enough to exclude the exterior stragling rays. But in smaller lenses where apertures are necessary, we have shewn that to preserve the same degree of distinctness their diameters must be as their focal distances; and then the apparent brightness will decrease in a duplicate ratio of their focal distances, so that by using smaller glasses the apparent magnitude and the obscurity of the object will both increase in the same ratio. For the ratio of PD to PF being invariable, the angle PFD is also invariable, and consequently the quantity of light received from the point F is also invariable; because the apertures of the lenses whether smaller or larger must all be situated at such distances from F, as just to receive all the rays contained in a cone described by turning the angle PFD about the axis PF, neither more Fig. 549.

 nor

nor less. But the apparent magnitude of the object, or the surface of its picture upon the retina is reciprocally as PF square [a], and consequently the light being the same its brightness is directly as PF square. By this theory it appears that a minute object cannot be magnified to infinity by a single lens though it were possible to make it as small as we please; without some method of increasing its light. Nevertheless this imperfection in single microscopes is not so great as at first sight one would take it to be, or as in fact we find it; the reason may be because the eye is capable of discerning objects tolerably well by above 20 thousand different degrees of light, every degree being equal to the light by which we see objects in the brightest moon-light night [b]. But though the brightness of the object were increased by throwing new light upon it, yet *Huygens* observes that the power of the microscope would still be limited by the breadths of the pencils that enter the pupil, which is equal to the breadth of the aperture. For if this breadth be less than $\frac{1}{5}$ or $\frac{1}{6}$ of a line, he affirms that the edges of the object will begin to appear indistinct [c]. But by double microscopes this excellent Author has made it appear we may magnify objects at pleasure; provided it was possible to form their object-glasses as small as we please; for we shall shew in the sequel that all other obstructions may be removed.

a Art. 118.
b Art. 95.
c Art. 358.

Proposition VI.

696. *In refracting and reflecting microscopes, and telescopes, made with a single eye-glass, the apparent indistinctness of a given object, caused by the aberrations of either kind considered separately, will be directly as the square of the greatest lateral aberration in the image formed by the object-glass or object-metal, and inversely as the square of the focal distance of the eye-glass, very nearly; because the aberrations caused by the eye-glass are almost inconsiderable.*

697. *Case* 1. Let us suppose the figures of the convex glasses EI, ei, to be perfect, or such as would cause no aberrations of homogeneal rays [d]; and let all the mean refrangible rays that flow from the point Q, in the axis $QEqex$, be refracted through the object-glass EI to the focus q; where crossing one another let them be refracted through the eye-glass ei into parallel lines; and let the humours of the eye be adapted to collect them accurately to a single point x upon the retina $x\varrho$. Then let the ray IRi be the outmost violet, or the outmost red, that was contained in the heterogeneal ray QI before refraction at I; and let this ray Ii be refracted by the eye-glass into the line ia, and be sent by the refractions in the eye to the point ϱ upon the retina. Lastly imagine a ray bi of the same colour as ai to come backwards parallel to the axis, and to be refracted at i along the line ir; and let a perpendicular to the axis through q, cut the ray ir

Fig. 550, 551.
d Art. 81.

in

in r and the ray iI in R. Now the semidiameter $\varkappa\varrho$, of the circle of aberrations upon the retina, is as the angle aib * or as its equal Rir * and consequently as $\frac{Rr}{Ri}$ * or nearly as $\frac{Rq}{qe}$. For I shall shew presently that qr is almost inconsiderable in comparison to qR in all optical instruments, notwithstanding it is not as qR. But the apparent indistinctness of the point Q, is as the area of the circle of aberrations upon the retina [a], or as the square of its semidiameter $\varkappa\varrho$ or as $\frac{qR^2}{qe^2}$; that is directly as the square of the lateral aberration in the image at q, and inversely as the square of the focal distance of the eye-glass. * Art. 688. * Art. 674. * Art. 222. a Art. 689.

698. *Case* 2. Supposing the glasses EI, ei to remain in the same places as before, let us now consider the aberrations which the sphericalness of their figures would cause in a pencil of mean refrangible rays. Therefore now let qR and $\varkappa\varrho$ represent the lateral aberrations of the outermost of these rays $QIRia\varrho$; and let bir be also a mean refrangible ray. Then $\varkappa\varrho$ is as the angle aib * or as Rir * or as $\frac{Rr}{Ri}$ * or as $\frac{Rq}{qe}$ neglecting qr. But the apparent indistinctness is as the circle of these aberrations at $\varkappa$ *, that is as $\varkappa\varrho^2$ or as $\frac{qR^2}{qe^2}$, that is, directly as the square of the lateral aberration in the image at q and inversely as the square of the focal distance of the eye-glass; as in the former case. Fig. 550. * Art. 688. * Art. 674. * Art. 222. * Art. 689.

699. To shew that qr, the aberration by the eye-glass, may be neglected, let EF be the focal distance of the object-glass of the double microscope; and let the ray Ii cross the axis in K; then in case 1. we have $qR = \frac{1}{55}\frac{Eq}{EF}EI$ *, and $qr = \frac{1}{55}ei = \frac{1}{55}\frac{eq}{Eq}EI$; because $ei : EI :: eK : KE :: eq : qE$ nearly. Consequently we have $qR : qr :: \frac{Eq^2}{EF} : eq$; that is as the apparent magnitude of the object Q seen in the microscope, to its apparent magnitude seen by the naked eye from a distance equal to Qq; as will appear by article 127. which gives some idea of the ratio of qR to qr. In *Huygens*'s standard microscope, to be described in art. 710, this ratio is as 35 to 1; and in a telescope it is also the ratio of its magnifying power. Fig. 550, 551. * Art. 678.

700. In case 2. considering the aberrations by the figure, we have $qR = \frac{7}{6}\frac{EI^3}{EF^3}Eq$ * and $qr = \frac{7}{6}\frac{ei^3}{eq^2} = \frac{7}{6}\frac{EI^3}{Eq^3}eq$; for the reason abovementioned. Consequently when the glasses E, e have similar shapes, we have $qR : qr :: \frac{Eq^4}{EF^3} : eq :: 3500 : 1$ in *Huygens*'s microscope. * Art. 681.

701. *Case* 3. Let us now consider reflecting microscopes and telescopes, in which let Q be the focus of incident rays upon a concave metal EI, whose semidiameter is EC and principal focus F; and let q be the conjuga- Fig. 552.

gate focus to $\mathfrak{Q}$; and $IKRe$ be a reflected ray making the aberration qK in length and qR in breadth; and let every thing else remain as in case 2. Then if we neglect the aberration qr caused by the refraction of the rays through the eye-glass ei; it is evident that the apparent indistinctness will be as $\frac{qR^2}{qe^2}$, as before in the refracting instruments.

702. To compare these aberrations qR and qr; we have the longitudinal aberration $qK = \frac{EI^2 \times qC^2}{EC^3}$. For by art. 648, we have $qK : \frac{1}{4}\frac{EI^2}{EC} :: qC^2 : CF^2$ or $\frac{1}{4}EC^2$. Hence the lateral aberration $qR = \frac{EI^3}{EC^3} \times \frac{qC^2}{qE}$. Now in the eye-glass, if we consider the aberrations caused by coloured

* Art. 699. rays, we had $qr = \frac{1}{55}\frac{eq}{qE} \times EI$*, and by consequence $qR : qr :: \frac{EI^2 \times qC^2}{EC^3} :$

a Art. 738. $\frac{1}{55}eq$; which in a reflecting microscope hereafter described[a] is as 53 to 1. By supposing $E\mathfrak{Q}$ to be infinitely increased, this microscope will be changed into Sir *Isaac Newton*'s reflecting telescope; and then we have $qR : qr :: \frac{EI^2 \times EF^2}{EC^3}$ or $\frac{EI^2}{8EF} : \frac{1}{55}eq :: 55 : 24$, in Mr. *Hadley*'s 5 foot tele-

b Art. 362. scope taking the middle eye-glass and aperture[b]. Nevertheless experience shews that the object appears sufficiently distinct. Indeed the dispropor-

* Art. 700. tion of qR to qr will be greater if we neglect the darker and fainter colours into which the ray ia is separated, which may scarce affect the eye, and take $qr = \frac{1}{250}ei$, supposing qe the focal distance of the brightest yel-

c Newt. Opt. p. 86. low[c]. But in refracting instruments the ratio of qR to qr will not be al-

d Art. 699. tered thereby[d]. With respect to the aberrations caused by the spherical

* Art. 700. figure of the eye-glass, we have $qr = \frac{7}{6}\frac{EI^3}{Eq^3}eq$*, and by consequence $qR : qr :: \frac{qC^2 \times qE^2}{EC^3} : \frac{7}{6}qe :: 7695 : 1$, in the reflecting microscope hereafter described. In a reflecting telescope this ratio is compounded of the ratio of its magnifying power and of 1 to $9\frac{1}{3}$ the eye-glass being plano-convex.

703. *Corol.* Therefore in refracting and reflecting telescopes and double microscopes, the apparent indistinctness of a given object, that would arise from the aberrations of either kind separately considered, will be nearly the same or invariable, when the focal distances of their eye-glasses are as the greatest lateral aberration in the images formed by their object-glasses or object-metals: or when the angle subtended by qR at the point i or e is invariable.

Definition. 704. And this angle is called the Angle of Aberration by the Figure or by Colours according as qR is the lateral aberration caused by the sphericalness of the figure or by coloured rays.

The

The following propositions being wrote when I read over Mr. *Huygens* upon the same subject[a], I followed the letters of his figures, and cannot easily change them for such as I commonly use without some danger of mistakes.

a Dioptr. p. 233.

LEMMA VI.

Concerning the apparent magnitude, the apparent brightness, and the angles of aberration, in double microscopes composed of two convex glasses.

705. Let an object BX be placed a little farther from the object-glass PD than its principal focus O; and let its image NY be viewed through an eye-glass EZ whose focal distance is NE. Let the object BX be viewed also by the naked eye from any given distance $B\Omega$; then take BQ to BO as EN to OP, and the object will be magnified in the microscope in the ratio of $B\Omega$ to BQ. Fig. 553.

For let the ray $XPYZ$ cut the image in Y, and be refracted by the eye-glass along ZV; join $X\Omega$ and draw XQ parallel to ZV or EY* and the object will appear from V under the angle EVZ equal to NEY or BQX; and consequently will be magnified in the ratio of the angle BQX to $B\Omega X$, or of $B\Omega$ to BQ. But since the figures PXQ, PYE are similar, we have $BQ : NE :: (BX : NY :: BP : PN ::) BO : OP$. For since B and N are conjugate focuses, we have $BO : BP :: BP : BN$* and disjointly $BO : OP :: BP : PN$. Q. E. D.

* Art. 50.

* Art. 239.

706. *Corol.* 1. The apparent distance $BQ = \frac{BP}{PN} NE = \frac{BO}{OP} NE$. And $B\Omega$ being given, the apparent magnitude of the object in the microscope, is reciprocally as BQ, by the lemma.

707. *Corol.* 2. Let PD be the semidiameter of the aperture of the object-glass, and the apparent brightness of the same object seen in the same or in different microscopes, will be as $\frac{PD^2 \times NE^2}{PN^2}$. For the quantity of rays that illuminate any particle of its picture upon the retina, is as $\frac{PD^2}{PB^2}$; because if PB was given, the quantity of rays received from B upon the whole aperture, would be as the aperture or as PD^2; and if PD was given, the quantity of those rays would be as their density in the aperture, that is as $\frac{1}{PB^2}$ *; but the apparent brightness of an object is directly as the quantity of rays that illuminate each particle of its picture upon the retina, and inversely as the area of the picture, or inversely as the apparent magnitude of the visible area of the object; and so the apparent brightness is directly as $\frac{PD^2}{PB^2} \times \frac{PB^2 \times NE^2}{PN^2}$ *.

* Art. 58.

* Art. 705.

708.

708. *Corol.* 3. The angle of aberration by colours is as $\frac{PD}{NE} \times \frac{PN}{PO}$ or as $\frac{PD}{NE} \times \frac{BP}{BO}$. For the greatest lateral aberration by colours in the image

* Art. 684. at N is as $PD \times \frac{PN}{PO}$*.

709. *Corol.* 4. The angle of aberration by the figure of the object-glass is as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$. For the greatest lateral aberration by the figure in the

a Art. 683. image at N is as $\frac{PD^3}{PO^3} PN$[a].

Proposition VII.

Fig. 553. 554. 710. *To make a new refracting microscope denoted by small letters* enpdbo, *that shall magnify an object more than a given microscope denoted by larger letters* ENPDBO, *in any proposed ratio of* n *to* 1; *with the same degree of brightness, and distinctness too, so far as it depends upon the different refrangibility of rays and not upon the sphericalness of the figure of the object-glasses.*

Take $ne = \frac{1}{n} NE$, $pd = \frac{1}{n} PD$; $po = \frac{1}{nn} PO$, $pb = \frac{1}{nn} PB$, $pn = \frac{1}{nn} PN$; and we have the microscope required. For example, the dimensions of *Huygens*'s standard microscope in inches were these; $NE = 2$, $PD = \frac{1}{20}$, $PO = \frac{1}{10}$, $PB = \frac{1}{9}$, $PN = 7$. And therefore the diameter of an object appeared in it 36 times longer than to the naked eye at 8 inches

b Art. 705. distance. Now to find the dimensions of another microscope that shall magnify twice as much, we have $n = 2$. Whence by the rule $ne = \frac{1}{2} NE = 1$, $pd = \frac{1}{40}$, $po = \frac{1}{40}$, $pb = \frac{1}{36}$, $pn = \frac{7}{4}$.

The rule is grounded upon this Hypothesis, of keeping the intervals of the points B, O, P, N, belonging to the object-glass, in the same given

c Art. 706. ratios. Hence we have NE reciprocally as the apparent magnitude[b] that is $ne : NE :: 1 : n$. Therefore $ne = \frac{1}{n} NE$. And because the angle of ab-

d Art. 703. errations by colours must remain unaltered[d], we have also PD as NE* or

* Art. 708. reciprocally as the apparent magnitude as before; that is $pd : PD :: 1 : n$. Therefore $pd = \frac{1}{n} PD$. And lastly because the apparent brightness must

* Art. 707. be kept unaltered, we have PN as $PD \times NE$* or as NE^2, (because we had PD as NE,) or reciprocally in a duplicate ratio of the apparent mag-

* By Hypoth. nitude; that is $pn : PN$ and $pb : PB$* and $po : PO$* $:: 1 : nn$. Therefore $po = \frac{1}{nn} PO$; $pb = \frac{1}{nn} PB$ and $pn = \frac{1}{nn} PN$. Q. E. D.

711. *Corol.* 1. In these microscopes the angles of aberration by the figure of their object-glasses are directly in a duplicate ratio of the appa-

rent

rent magnitudes of the object. For these angles are as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$ * or as $\frac{PD^2}{PO^2}$ by the hypothesis of the rule and because we had PD as NE; that is, the angle of this aberration in the new microscope, is to the like angle in the old, as $\frac{pd^2}{po^2}$ or $\frac{nnPD^2}{PO^2}$ to $\frac{PD^2}{PO^2}$, that is, as nn to 1. * Art. 709.

712. *Corol.* 2. Hence if a given microscope that has a plano-convex object-glass, will bear the convex side to be turned towards the object, which increases the angle of aberration by its figure near 4 times[a]; by substituting for it a new plano-convex glass, with its plane side toward the object, whose focal distance is 4 times shorter, the angle of aberration will also become quadruple[b], because the apparent magnitude of the object will be doubled[c]. Mr. *Huygens* found that his microscope would bear this inversion. But if we try to magnify much more by this proposition, the aberrations by the figure will still increase and put a stop to this process; which nevertheless might be continued to infinity by the following proposition, as this excellent Author has observed, but for the practical difficulty of making the object-glasses so small as are requisite for that purpose.

a Art. 661. 662. 681.
b Art. 711.
c Art. 710.

PROPOSITION VIII.

713. *To make a new refracting microscope denoted by small letters* enpdbo, *which shall magnify an object more than a given microscope denoted by larger letters* ENPDBO, *in any proposed ratio of* n *to* 1, *with the same brightness, and distinctness too, with respect to the aberrations caused by the figure; and with greater distinctness with respect to the aberrations caused by colours.* Fig. 553, 554.

Take $ne = \frac{1}{n} NE$, $pd = \frac{1}{n^3} PD$, $po = \frac{1}{n^4} PO$, $pb = \frac{1}{n^4} PB$, $pn = \frac{1}{n^4} PN$; and you have the dimensions for the purpose required; and the angle of aberration caused by colours will be less in this than in the given microscope, in the ratio of 1 to nn. For example in the standard microscope abovementioned, putting $n=2$, we have $ne=1$, $pd=\frac{1}{16}$, $po=\frac{1}{16}$, $pb=\frac{7}{144}$, $pn=\frac{1}{16}$; and the angle of aberration by colours 4 times less than before.

This rule is grounded upon the same hypothesis as the former, *viz.* that the ratios of the intervals of the points B, O, P, N, are the same in both microscopes. Therefore since the angles of aberration by the figure must be the same in both[d], we have PD^3 as $PO^2 \times NE$*; and since the brightness must be the same too, we have PN^2 or PO^2* as $PD^2 \times NE^2$*, and consequently PD^2 as $\frac{PO^3}{NE^3}$. Substitute these values of PO^2

d Art. 703.
* Art. 709.
* Hypoth.
* Art. 707.

and

and PD^3 in the foregoing system (PD^3 as $PO^2 \times NE$) and we have PD as NE^3, and PO as NE^4. But NE being reciprocally as the apparent magnitude[a], or $ne : NE :: 1 : n$, we have $ne = \frac{1}{n} NE$; and PD being as NE^3, or $pd : PD :: ne^3 : NE^3 :: 1 : n^3$, we have $pd = \frac{1}{n^3} PD$; and in like manner PO being as NE^4, we have $po = \frac{1}{n^4} PO$; and by the hypothesis pb and pn are as po.

[a] Art. 706.

Now the angle of aberration by colours is as $\frac{PD}{NE}$ *; that is this angle in the new microscope is to the like angle in the old, as $\frac{pd}{ne}$ or $\frac{PD}{nnNE}$ to $\frac{PD}{NE}$, that is, as 1 to nn. Q. E. D.

* Art. 708.

714. *Corol.* The breadth of the pencils that enter the pupil are also the same*, for half this breadth is $EI = \frac{PD \times NE}{PN}$; so that by this proposition we might magnify to infinity without any impediments, but from the minuteness of the object-glass. But by the following propositions which do the same thing, the object-glass is not diminished in so great a proportion as by this present one; being not restrained by the given ratios of the intervals of the points B, O, P, N.

* Art. 707.

PROPOSITION IX.

Fig. 553, 554.

715. *If it be required to compose a microscope of two convex lenses* e *and* p, *which with the given eye-glass* e *shall magnify in a given ratio, and in which the apparent brightness of the object, and the angle of aberration by colours, shall be the same as in another given microscope composed of two lenses* E *and* P; *the focal distance of the object-glass* p *and its aperture and position may be found in this manner.*

Supposing the same schemes as before; let the given dimensions of the old microscope be these; $PD = A$, $PO = C$, $NE = D$, $BO : BP :: 1 : m$. And suppose the corresponding dimensions of the new microscope to be these; $pd = a$, $po = c$, $ne = d$, $bo : bq :: 1 : n$; and let the apparent magnitude proposed, be to the apparent magnitude seen by the naked eye from a given distance ω, in a given ratio of ω to q. Then we shall have $c = \overline{\frac{md}{D} \times \frac{q}{d+q}}\Big|^2 \times C$; $n = \frac{md}{D} \sqrt{\frac{C}{c}}$; and $a = A \sqrt{\frac{c}{C}}$.

For since the apparent brightness must be the same in both microscopes, we must put $\frac{PD \times NE}{PN} = \frac{pd \times nc}{pn}$*; that is $\frac{AD}{mC} = \frac{ad}{nc} = f$, to abridge the following reduction. And since the angle of aberration by colours must also be

* Art. 707.

be the ſame in both, we muſt put $\frac{PD}{NE} \times \frac{BP}{BO} = \frac{pd}{ne} \times \frac{bp}{bo}$ *; that is $m\frac{A}{D} =$ * Art. 708.
$n\frac{a}{d} = g$. By the former of theſe equations we have $\frac{ncf}{d} = a = \frac{dg}{n}$ by the latter. Hence $nn = \frac{dd}{c} \times \frac{g}{f} = \frac{dd}{c} \times \frac{mm}{DD}C$, and $n = m\frac{d}{D}\sqrt{\frac{C}{c}}$; and by putting this value for n, we have $a = A\sqrt{\frac{c}{C}}$. Now $q = \frac{ob}{op}ne$ * $= \frac{d}{n-1}$; and * Art. 706.
thence $\frac{d+q}{q} = n = m\frac{d}{D}\sqrt{\frac{C}{c}}$; and $\overline{\frac{d+q}{q}}\Big|^2 = \overline{\frac{md}{D}}\Big|^2 \times \frac{C}{c}$; and $c = \overline{\frac{md}{D}}\Big|^2 \times \overline{\frac{q}{d+q}}\Big|^2 C$.

716. *Corol.* 1. Hence in theſe microſcopes the diameters of the apertures of the object-glaſſes are in a ſubduplicate ratio of their focal diſtances, as in common teleſcopes. For we had $\frac{a}{A} = \sqrt{\frac{c}{C}}$. And when theſe microſcopes are changed into teleſcopes by making BO and bo infinite, and conſequently $m = 1$ and $n = 1$, we have alſo $\frac{d}{D} = \sqrt{\frac{c}{C}} = \frac{a}{A}$; which agrees with art. 354.

717. *Corol.* 2. In theſe microſcopes the angles of aberration by the figure of the object-glaſſes, are reciprocally as their focal diſtances; and by conſequence are reciprocally in a duplicate ratio of their apertures[a]. a Art. 716.
For the angle of aberration by the figure is as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$ * $= \frac{mA^3}{CCD}$; that is * Art. 709.
this angle in the old microſcope, is to the like angle in the new as $\frac{mA^3}{CCD}$ to $\frac{na^3}{ccd}$, or as $\frac{AA}{CC}$ to $\frac{aa}{cc}$ (becauſe we had $\frac{mA}{D} = \frac{na}{d}$ *,) that is, as $\frac{1}{C}$ to $\frac{1}{c}$; be- * Art. 708.
cauſe $AA : aa :: C : c$ *. * Art. 716.

718. *Corol.* 3. By keeping the ſame object-glaſs and by altering the focal diſtance of the eye-glaſs, the apparent magnitude of the object can be increaſed but little. For by the value of $q = \frac{d}{n-1}$ *, we have $q : d :: 1$ * Art. 706.
$: n-1$ and $q+d : d :: n : n-1$; we have alſo $pb : po :: n : n-1$; therefore $pb = \frac{d+q}{d}c$, and $pb^2 = \overline{\frac{d+q}{d}}\Big|^2 \times c \times (c) \frac{mm}{DD} \times \overline{\frac{dq}{d+q}}\Big|^2 \times C = \frac{mm}{DD}cCqq$; and conſequently when c is given, we have pb as q. But pb muſt always be ſomewhat bigger than po; and therefore if at firſt it be put leſs than $2po$, as it muſt be, to make the image bigger than the object; it is manifeſt that pb and conſequently q cannot be diminiſhed ſo much as in the ratio of 2 to 1, that is the apparent magnitude cannot be doubled.

719. *Corol.* 4. But if we keep the ſame eye-glaſs and diminiſh the focal diſtance of the object-glaſs, we might magnify as much as we pleaſe but for the increaſe of the aberrations by the figure[b]. For ſince the value b Art. 717.

of po may be thus expressed, $c = \left(\frac{m}{D} \times \frac{d}{\frac{d}{q} + 1}\right)^2 \times C$; by increasing the magnifying power, that is by diminishing q, the denominator of this quantity will be increased and c will be diminished. Therefore to improve the given microscope as far as possible, we must retain much the same eye-glass and diminish the object-glass and its aperture according to the rules above, till we find the aberrations by its figure begin to be troublesome; and if it be required to magnify still more, we may from the last microscope, in which the aberrations by the figure do no harm, determine another by the following proposition.

Proposition X.

Fig. 553, 554. 720. *If it be required to compose a microscope of two convex lenses* e *and* p, *which with the given eye-glass* e *shall magnify in a given ratio; and in which the apparent brightness of the object, and the angle of aberration by the figure shall be the same as in another given microscope composed of two lenses* E *and* P, *the focal distance of the object glass* p *and its aperture and position may be found in this manner.*

Supposing the same schemes and notation as before, we shall have $c = \left(\frac{md}{D} \times \frac{q}{d+q}\right)^4 C$; $n = \frac{md}{D}\sqrt[4]{\frac{C}{c}}$; $a = A\sqrt[4]{\frac{c^3}{C^3}}$.

For since the apparent brightness must be the same in both micro-
* Art. 707. scopes, we must put $\frac{PD \times NE}{PN} = \frac{pd \times ne}{pn}$ *; that is $\frac{AD}{mC} = \frac{ad}{nc} = f$ as before. And since the angle of aberration by the figure must be the same in both,
* Art. 709. we must put $\frac{PD^3}{PO^3} \times \frac{PN}{NE} = \frac{pd^3}{po^3} \times \frac{pn}{ne}$ *; that is $\frac{mA^3}{CCD} = \frac{na^3}{ccd} = b$ to shorten the work. Hence we have $a = \frac{ncf}{d}$; and $\frac{n^3c^3f^3}{d^3} = (a^3 =) \frac{ccdb}{n}$; that is $n^4 = \frac{d^4}{c} \times \frac{b}{f^3} = \frac{d^4}{c} \times \frac{m^4}{D^4} C$; and $n = \frac{md}{D}\sqrt[4]{\frac{C}{c}}$. And thence $a = (\frac{ncf}{d} =) A$
* Art. 706. $\sqrt[4]{\frac{c^3}{C^3}}$. But $q = \frac{ob}{op} ne$* $= \frac{d}{n-1}$; and thence $\frac{d+q}{q} = (n =) \frac{md}{D}\sqrt[4]{\frac{C}{c}}$; and $\left(\frac{d+q}{q}\right)^4 = \left(\frac{md}{D}\right)^4 \times \frac{C}{c}$; and $c = \left(\frac{md}{D} \times \frac{q}{d+q}\right)^4 C$.

721. *Corol.* 1. In these microscopes the diameters of the apertures of the object-glasses, are as the biquadrate roots of the cubes of their focal distances. For we had $\frac{a}{A} = \sqrt[4]{\frac{c^3}{C^3}}$.

722. *Corol.* 2. And the angles of aberration by colours, caused by the object-glasses, are directly in a subduplicate ratio of their focal distances.
a Art. 717. The manner of proof is the same as before[a].

723, *Corol.* 3. Here alſo the magnifying power can be but little increaſed by changing the eye-glaſs. For we ſhall find pb as q, in like manner as before[a]. a Art. 718.

724. *Corol.* 4. In the former propoſition we had po as $\overline{\frac{d}{\frac{d}{q}+1}}\Big|^2$, and by the preſent propoſition po is as $\overline{\frac{d}{\frac{d}{q}+1}}\Big|^4$; ſo that for the ſame increaſe of the magnifying power, or decreaſe of q, the preſent po decreaſes in a duplicate ratio of the former. For which reaſon it is beſt to magnify as much as we can by the former propoſition before we apply the latter; that the object-glaſs may be preſerved as large as poſſible. And by this microſcope we might magnify to infinity but for the ſmallneſs of the object-glaſs; becauſe the brightneſs will continue the ſame and the diſtinctneſs will be increaſed by cor. 2.

725. *Corol* 5. If a new microſcope be required in which the angles of aberration of both kinds ſhall be reſpectively the ſame as in the old one; we muſt retain the ſame object-glaſs; and to magnify more we muſt increaſe the focal diſtance of the eye-glaſs; but if it be increaſed to infinity the apparent magnitude in the new microſcope, will be to the apparent magnitude in the old, but as m to $m-1$, or in *Huygens*'s ſtandard but as 10 to 9, which is but a trifle. For the angle of aberration by the figure will be the ſame by putting $\frac{mA^3}{CCD}=\frac{na^3}{ccd}$*, in which by ſubſtituting the values of n and a found in the former propoſition, which ſuppoſed the angle of aberration by colours to be the ſame, we ſhall find $c=C$; therefore in the value of $c=\frac{md}{D}\times\overline{\frac{q}{d+q}}\Big|^2 C$*; we muſt put $\frac{md}{D}\times\frac{q}{d+q}=1$; which gives $d=\frac{1}{\frac{m}{D}-\frac{1}{q}}$; by which it appears that d will increaſe by diminiſhing q; and will become infinite when $\frac{m}{D}-\frac{1}{q}=0$, or when $q=\frac{D}{m}$; but in the old microſcope $Q=\frac{D}{m-1}$*; therefore we have Q to q, or the apparent magnitude in the new, to the apparent magnitude in the old, as m to $m-1$. It is to no purpoſe therefore to alter the eye-glaſs as we ſhewed before.

* Art. 709.

* Art. 715.

* Art. 706.

726. To give an example of each of theſe propoſitions; in *Huygens*'s ſtandard microſcope we have $NE=D=2$; $PD=A=\frac{1}{20}$; $PO=C=\frac{3}{10}$; $PB=\frac{2}{9}$; whence $m=10$ and $Q=\frac{D}{m-1}$ * $=\frac{2}{9}$. To magnify as much more, * Art. 706.

* Art. 706. more, we must put $q=\frac{1}{9}$*. Hence by the former proposition putting $d=D=2$ we have $po=c=\frac{70}{361}$; $n=19$; $pb=\frac{35}{171}$; and $pd=a=\frac{1}{38}$: But by this proposition $po=c=\frac{7000}{\overline{19}|^4}=\frac{1}{19}$ nearly, and the rest according to the rules. If we put $d=1$, by the former proposition we shall have $po=\frac{7}{40}$; and by the present proposition $po=\frac{7}{160}$; which agree with the rules in the 710 and 713 articles.

Lemma VII.

727. Concerning the apparent magnitude, brightness and distinctness of objects in reflecting microscopes, composed of a reflecting concave-metal and a single eye-glass.

Fig. 555. Let an object BX be placed between the center P and principal focus T of a reflecting concave surface ACG; and let the image NY be viewed through the eye-glass EZ, whose focal distance is NE. Let the object BX be viewed also by the naked eye from any given distance $B\Omega$; then take BQ to BT as NE to TC, and the object will be magnified in the microscope in the ratio of $B\Omega$ to BQ.

For any line PX, produced both ways to the concave at G and to the eye-glass at Z, is the axis of an inclined pencil of rays, flowing from X, reflected from G to the image at Y, and refracted at Z to the eye at V.

* Art. 50. Draw XQ parallel to ZV or EY*, and the object will appear in the microscope under the angle EVZ equal to NEY or BQX; and consequently will be magnified in the ratio of the angle BQX to $B\Omega X$ or of $B\Omega$ to BQ. But $BQ : NE :: (BX : NY :: PB : PN ::) TB : TP$ or TC; be-

a Art. 207. because TB, TP, TN are continual proportionals[a]. Q. E. D.

728. *Corol.* 1. The apparent distance $BQ=\frac{TB}{TC}\times NE=\frac{PB}{PN}\times NE=\frac{CB}{CN}\times NE$: and $B\Omega$ being given, the apparent magnitude of the object is reciprocally as BQ, by the lemma.

729. *Corol.* 2. Let CA be the semiaperture of the reflecting concave, and the apparent brightness of the same object, in the same or in different microscopes, will be as $\frac{CA^2\times NE^2}{CN^2}$. For it is directly as the quantity of light received by the surface of the concave from any particle B, and inversely as the area of the picture of that particle upon the retina or inversely as the apparent magnitude of the area of the particle; and conse-

* Art. 58. * Art. 728. quently is as $\frac{CA^2}{CB^2}$ *$\times\frac{CB^2\times NE^2}{CN^2}$ *.

730. *Corol.* 3. The apparent indiſtinctneſs of a given object, in the ſame or in different reflecting microſcopes, is as $\frac{CA^6}{CT^6} \times \frac{PN^4}{CN^2 \times NE^2}$; and conſequently is invariable when $\frac{CA^3}{CT^3} \times \frac{PN^2}{CN \times NE}$ is invariable. For the greateſt lateral aberration in the picture at N is as $\frac{CA^3}{CT^3} \times \frac{PN^2}{NE}$ * and the apparent indiſtinctneſs is as the ſquare of this aberration directly and as NE^2 inverſely [a], neglecting the aberrations cauſed by the eye-glaſs according to art. 702.

* Art. 702.

a Art. 696.

PROPOSITION XI.

731. *A new reflecting microſcope denoted by ſmall letters* abcpnev, *may be made to magnify more than a given reflecting microſcope, adjuſted by experiments and denoted by larger letters* ABCPNEV, *in any given ratio of* n *to* 1, *with the ſame brightneſs and diſtinctneſs too pretty nearly, by taking* $ne = \frac{1}{n}NE$; $ca = \frac{1}{n^3}CA$; $ct = \frac{1}{n^4}CT$; $cb = \frac{1}{n^4}CB$; $cn = \frac{1}{n^4}CN$. Fig. 555, 556.

732. This propoſition may be demonſtrated by the foregoing lemma and its corollaries, in the ſame manner as the 8th propoſition was demonſtrated by its lemma. For here alſo the intervals of the points C, B, P, N are ſuppoſed to be in given ratios. But the following propoſition is more general and better than this, becauſe it will not diminiſh the focal diſtance of the concave ſo much as this does.

PROPOSITION XII.

733. *Having a reflecting microſcope conſiſting of a concave metal* CA *and a convex eye-glaſs* EZ *adjuſted together by experiment; it is propoſed to adjuſt any other given concave* ca *and convex eye-glaſs* ez, *ſo that the apparent brightneſs of the object ſhall continue the ſame as in the given microſcope, and the apparent diſtinctneſs too, neglecting the increment of the aberrations cauſed by the new eye-glaſs* ez *if it be taken leſs than* EZ: *and then to ſhew how much the new microſcope will magnify.* Fig. 555, 556.

In the 555th figure already deſcribed, let the dimenſions of the given microſcope be theſe; $CA = A$, $CT = C$, $NE = D$ and $TB : TC :: 1 : m$; and in the new microſcope, whoſe correſponding parts are denoted by ſmall letters of the ſame names, let the given lines $tc = c$, $ne = d$. Take a number $n = \sqrt{1 + mm} - 1, \frac{dd}{DD}\sqrt{\frac{C}{c}}$; and for the places of the object and of the eye-glaſs we ſhall have $tb : tc$ and $tc : tn :: 1 : n$; and the

ſemi-

ſemiaperture ca or $a = \frac{n+1}{m+1} \times \frac{cD}{Cd} A$: and the apparent magnitude of
the object in the new microſcope, will be to its apparent magnitude, ſeen by
* Art. 728. the naked eye, from a given diſtance ω, as ω to $\frac{d}{n}$* or $\frac{d}{\sqrt{\overline{1+\overline{mm-1},\frac{dd}{DD}\sqrt{\frac{C}{c}}}}}$.

For ſince the apparent brightneſs muſt be the ſame in both micro-
* Art. 729. ſcopes, we muſt put $\frac{CA\times NE}{CN} = \frac{ca\times ne}{cn}$*, that is $\frac{AD}{\overline{m+1},C} = \frac{ad}{\overline{n+1},c} = f$, to
ſhorten the following reduction. And ſince the apparent diſtinctneſs muſt
* Art. 730. be the ſame too, we muſt put $\frac{CA^3}{CT^3}\times\frac{PN^2}{CN\times NE} = \frac{ca^3}{ct^3}\times\frac{pn^2}{cn\times ne}$*; that is
$\frac{\overline{m-1}^2}{m+1}\times\frac{A^3}{CCD} = \frac{\overline{n-1}^2}{n+1}\times\frac{a^3}{ccd} = g$. By the former equation we have $a =$
$\frac{\overline{n+1},cf}{d}$, and by cubing $\frac{\overline{n+1}^3,c^3f^3}{d^3} = (a^3 =) \frac{n+1}{\overline{n-1}^2}\times ccdg$, by the latter
equation. Hence $\overline{nn-1}^2 = \frac{d^4 g}{cf^3}$; and $nn-1 = dd\sqrt{\frac{g}{cf^3}} = dd\times\frac{mm-1}{DD}\times$
$\sqrt{\frac{C}{c}}$, by reſtoring the values of f and g; and $n=\sqrt{\overline{1+\overline{mm-1}\,\frac{dd}{DD}\sqrt{\frac{C}{c}}}}$.
And by ſubſtituting the value of f in this equation $a=\frac{\overline{n+1},cf}{d}$, we have
the value of a as above. Since the breadth of the middle pencil, where it
a Art. 729. emerges from any eye-glaſs, is always the ſame[a]; if we retain the ſame eye-
714. glaſs, it will cauſe the ſame aberrations; but if we diminiſh the eye-glaſs,
by a greater refraction it will increaſe its own aberrations and conſequent-
b Art. 702. ly the apparent indiſtinctneſs. For though the bigger part[b] of the whole
Fig. 552. angle of aberration aib or Rir is kept invariable; yet the leſſer part of it,
which is ſubtended by qr, will vary reciprocally as qe, taking qr for the
aberration by colours; and reciprocally as qe cube, taking qr for the aber-
ration cauſed by the figure of the eye-glaſs. For this part of that angle
* Art. 222. being as $\frac{qr}{qe}$*, is as $\frac{ei}{eq}$ in the firſt caſe[c], and as $\frac{ei^3}{eq^3}$ in the ſecond caſe[d],
c Art. 327. d Art. 700. ſuppoſing we always apply ſimilar eye-glaſſes. *Q. E. D.*

734. *Corol.* 1. Hence when ω is given, the apparent magnitude in this microſcope is as $\frac{n}{d}$ or as $\sqrt{\frac{1}{dd}+\frac{mm-1}{DD}\sqrt{\frac{C}{c}}}$; and conſequently may be increaſed at pleaſure by diminiſhing c; and alſo by diminiſhing d, if we neglect the ſmall increment of the aberrations cauſed by ſmaller eye-glaſſes.

735. *Corol.* 2. Hence putting $q = \frac{d}{n}$ the apparent distance of the object[a]; if c and q be given and the rest be required, we shall have $d = \frac{qD}{\sqrt{DD + \overline{1-mm},qq\sqrt{\frac{C}{c}}}}$; and consequently $n = \frac{d}{q}$, and $tb:tc::1:n$, and $a = \frac{n+1}{m+1} \times \frac{cD}{Cd}A$. For since $q = (\frac{d}{n} =) \frac{d}{\sqrt{1 + \overline{mm-1},\frac{dd}{DD}\sqrt{\frac{C}{c}}}}$, by reduction we shall find d as above. If the old concave be retained we shall have $d = \frac{D}{\sqrt{\frac{DD}{qq} + 1 - mm}}$.

[a] Art. 728.

736. *Corol.* 3. If d and q be given and the rest be required, we shall have $c = \left(\overline{\frac{mm-1}{DD} \times \frac{ddqq}{dd-qq}}\right)^2 \times C$, by reduction as before; and tb and ca the same as in corol. 2. If the old eye-glass be retained, we shall have $c = \left(\overline{mm-1} \times \frac{qq}{DD-qq}\right)^2 \times C$, and the apparent distinctness will continue the very same as in the given microscope.

737. *Corol.* 4. This microscope may be changed into Sir *Isaac Newton*'s reflecting telescope by making TB and tb infinite, and consequently $m=0$ and $n=0$, that is $1 - \frac{dd}{DD}\sqrt{\frac{C}{c}} = 0$. Hence $\frac{D}{d} = \sqrt[4]{\frac{C}{c}}$ or $d:D::\sqrt[4]{c}:\sqrt[4]{C}$. We have also $\frac{a}{A} = \frac{cD}{Cd} = \frac{c}{C}\sqrt[4]{\frac{C}{c}} = \sqrt[4]{\frac{c^3}{C^3}}$ that is $a:A::\sqrt[4]{c^3}:\sqrt[4]{C^3}$. And because $\frac{C}{D}a = \frac{c}{d}A$, we have $\frac{c}{d}:\frac{C}{D}::(a:A::)\sqrt[4]{c^3}:\sqrt[4]{C^3}$, or the magnifying powers as $\sqrt[4]{C^3}$. All which agree with art. 361.

738. Having made a few gross experiments with a concave metal that I had by me, whose focal distance was $\frac{2}{3}$ inch, and with several convex eye-glasses applied to it; I found that the colours of objects in a reflecting microscope appeared much more beautiful and natural than in double refracting microscopes of the best sort; their proper colours being free from the mixture of other colours arising in refracting microscopes from the different refrangibility of rays. When some small hairs and a mite were placed upon a plane piece of glass at B, and had scarce any other illumination than the direct sky-light at a window, I found they appeared sufficiently bright and very distinct when the dimensions of my reflecting microscope were these in parts of an inch; CA or $A = \frac{1}{2}$; CT or $C = \frac{2}{3}$; NE or $D = \frac{2}{3}$; $TB:TC::1:14$. Therefore we have $m = 14$, and the

Dimensions of a reflecting microscope.

appa-

* Art. 728. apparent distance $BQ = \frac{D}{m}* = \frac{1}{6}$; and consequently, putting $B\Omega = 8$ inches, these objects were magnified 48 times in diameter.

739. Hence it is easy to compute from corol. 3, that to magnify 72 times with the same eye-glass of $\frac{1}{3}$ inch focal distance, and consequently with the very same distinctness and brightness as in this standard experiment, the focal distance ct of the new concave must be 0, 458 of an inch; and consequently the diameter of the spherical surface of which this concave is a portion, that is $4\,ct$, is 1, 832. Which is above 9 times bigger than $po = \frac{70}{361} = 0, 194$, the focal distance and diameter of the sphere of which that plano-convex lens is a portion, which we found would also magnify 72 times with an eye-glass of 2 inches focal distance[a]; but not so distinctly as its standard did; because the angle of aberration by the spherical figure of the object-glass was increased almost 4 times[b]. Now this excess of the diameter of the sphere of the concave metal above that of the lens, is a considerable advantage in order to magnify still farther by a farminution of the concave.

a Art. 726.

b Art. 717.

740. But how far the magnifying power of a given concave may be successfully promoted by diminishing the eye-glass according to the rule in corol. 2, I could not examine experimentally for want of proper mechanism, to adjust the intervals between the concave, the object and the eye-glass, according to computation; and also to hold a convex lens or a reflecting concave in a proper position for casting light upon the object; and to compute it by theory would be troublesome.

741. These I suppose may be the advantages which Sir *Isaac Newton* expected, when he tells us he sometimes had thoughts of making a microscope of this sort[c]. Nevertheless in attempting to magnify very much we shall soon be stopt here too, by the minuteness of the concaves that are necessary for the purpose. This put me upon contriving a microscope with two reflecting spherical surfaces of any size, so proportioned to each other that the aberrations of the rays caused by the first reflection, shall be perfectly corrected by the second; and by consequence that the last image of the object, from which the rays diverge upon the eye-glass, shall be as perfectly free from aberrations, as the theory of these aberrations is perfect. But having made the demonstration of this construction quite independent upon any thing contained in this troublesome chapter, I chuse to give it as a Remark.

c Phil. Trans. N°. 80.

A COM-

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